VISTA

FOCAL PROPERTIES OF OPTICAL AND ELECTROMAGNETIC SYSTEMS*

J. L. SYNGE, Ohio State University

1. Preface. This is a Preface, not an Introduction. Like the Prefaces to the plays of George Bernard Shaw, it has only the remotest connection with what follows, and may be read first, last, or not at all. Anyone who reads it must form his own judgment regarding its truth insofar as it deals with the past and the present. As for the future, who knows?

Most mathematicians work with ideas which, by common consent, belong definitely to mathematics. They form a closed guild. The initiate foreswears the things of the world, and generally keeps his oath. Only a few mathematicians roam abroad and seek mathematical sustenance in problems arising directly out of other fields of science. In 1744 or 1844 this second class included almost the whole body of mathematicians. In 1944 it is such a small fraction of the whole that it becomes necessary to remind the majority of the existence of the minority, and to explain its point of view.

The minority does not wish to be labelled "physicist" or "engineer," for it is following a mathematical tradition extending through more than twenty centuries and including the names Euclid, Archimedes, Newton, Lagrange, Hamilton, Gauss, Poincaré. The minority does not wish to belittle in any way the work of the majority, but it does fear that a mathematics which feeds solely on itself will in time exhaust its interest. This attitude has been summarized recently by Birkhoff [1]: "It will probably be the new mathematical discoveries which are suggested through physics that will always be most important, for, from the beginning, Nature has led the way and established the pattern which mathematics, the language of Nature, must follow."

Apart from its effect on the future of mathematics proper, the isolation of mathematicians has robbed the rest of science of a support on which it counted in all previous epochs. It is true that an immense amount of mathematical work now appears in physical and engineering journals. The technical scientist of today knows a great deal more mathematics than his grandfather knew. But does anyone seriously think that the role played in the past by the greatest mathematicians of their day can be taken over by technical scientists to whom mathematics is merely a tool? Of course they cannot take over that role, and they would be the first to admit it. Out of the study of nature there have originated (and in all probability will continue to originate) problems far more difficult than those constructed by mathematicians within the circle of their own ideas. Scientists have relied on the mathematician to throw his energies against these problems. They know that the mathematician is not merely a skilled user of certain tools already made—they can use those tools with no inconsiderable skill themselves. Rather they look to the qualities peculiar to the mathemati-

* Presented at the meeting of the Mathematical Association of America in Chicago, November, 1943.

cian—his logical penetration and capacity to see the general in the particular and the particular in the general.

Through three centuries a magnificant cooperation raised our civilization to an intellectual height never before achieved by man. We have been far sharper than the Greeks, far more practical than the Romans. Whatever the historian of the future may say about the follies and vices of these three centuries—their wars, robberies and greeds,—he will at least say: "This was a great age of science!"

In all this the mathematician was the directing and disciplining force. He gave science its methods of calculation—logarithms, calculus, differential equations, and so on—but he gave it much more than this. He gave it a blue-print. He insisted that thought be logical. As each new science came up, he gave it—or tried to give it—the firm logical structure that Euclid gave to Egyptian land surveying. A subject came to his hands a rough stone, trailing irrelevant weeds; it left his hands a polished gem.

At present science is humming as it never hummed before. There are no obvious signs of decay. Only the most observant have noticed that the watchman has gone off duty. He has not gone to sleep. He is working as hard as ever, but now he is working solely for himself. In his present attitude towards the rest of science there may be a shade of resentment. Theoretical physicists stole the public applause, and tried to use him as a mere calculator. However that may be, the fact is that the brain of science is no longer where it used to be. It is not dead or out of order; it has simply been removed by a painless operation and set up apart from the body it so long directed.

In brief, the party is over—it was exciting while it lasted. I do not share Birkhoff's optimism. Nature will throw out mighty problems, but they will never reach the mathematician. He may sit in his ivory tower waiting for the enemy with an arsenal of guns, but the enemy will never come to him. Nature does not offer her problems ready formulated. They must be dug up with pick and shovel, and he who will not soil his hands will never see them.

Change and death in the world of ideas are as inevitable as change and death in human affairs. It is certainly not the part of a truth-loving mathematician to pretend that they are not occurring when they are. It is impossible to stimulate artificially the deep sources of intellectual motivation. Something catches the imagination or it does not, and, if it does not, there is no fire. If mathematicians have really lost their old universal touch—if, in fact, they see the hand of God more truly in the refinement of precise logic than in the motion of the stars—then any attempt to lure them back to their old haunts would not only be useless—it would be denial of the right of the individual to intellectual freedom. But each young mathematician who formulates his own philosophy—and all do—should make his decision in full possession of the facts. He should realize that if he follows the pattern of modern mathematics he is heir to a great tradition, but only part heir. The rest of the legacy will have gone to other hands, and he will never get it back.

A man who spends his time deploring the loss of a vanished glory is a nuisance and a bore. If, indeed, the mathematician of the future is to use his finest efforts within the circle of purely mathematical ideas, let us make plans for the future with that fact admitted as an essential premise. I do not mean plans for mathematics itself, but plans for the future relationship of mathematics to the rest of science. Just now we are in an awkward and slightly embittered period. The awkwardness and bitterness arise from a failure on both sides to realize what has taken place. The non-mathematical scientist does not know that the mathematician has given up his general supervision of science, and is embittered by the mathematician's lack of interest. The mathematician does not realize that he has lost the leadership, and is embittered by the fact that his finest creative efforts carry so little weight in the body of science as a whole.

What we all need is more mutual understanding and respect. I do not mean just refraining from overt criticism, but a genuine and humble attempt to understand the motivation and thought-processes of other men of science. If this is achieved, there is a hope that, though the mathematician is no longer the great leader, he will remain the counsellor and friend of all science. It will not be the glory of the past, but at least a modus vivendi. Our science started with mathematics and will surely end not long after mathematics is withdrawn from it (if it is withdrawn). A century hence there will be bigger and better laboratories for the mass-production of facts. Whether these facts remain mere facts or become science will depend on the extent to which they are brought into contact with the spirit of mathematics.

2. The variational principle in optics. There are two basic laws in optics—the law of reflection and the law of refraction. They form the axioms or postulates of a mathematical theory—geometrical optics. Starting from them, the mathematician can study the behavior of light rays passing through an optical instrument (camera, telescope, or microscope) without any knowledge of physics. However, the wise mathematician will ask the physicist what questions would be most interesting to investigate, since otherwise his researches may wander through lack of a goal. One of the interesting problems—in fact, the main problem—is that of the formation of images.

The laws of reflection and refraction are extremely simple. A knowledge of high-school mathematics is sufficient to understand them. But mathematics at the high-school level is not always easy. The geometry of the point, line, and circle, for example, provides problems of great difficulty because there is no guiding principle for their solution. Workers in optics find the same need for a guiding principle.

It is a rule of mathematics that axioms should not be redundant. It should not be possible to prove one axiom from another. So, to get things straight, we wipe out the laws of reflection and refraction from our sheet of axioms, and write instead:

$$\delta \int nds = 0. \tag{1}$$

From this single axiom, known as Fermat's Principle, the whole of geometrical optics follows logically. The laws of reflection and refraction are reinstated, not as axioms but as deductions from equation (1). Actually, we have widened the scope of our theory by adopting (1), rather than the laws of reflection and refraction. We can use (1) to discuss situations with which the elementary laws cannot cope. In spite of this, and although Fermat's dates were 1601–1665, most physicists seem to regard Fermat's Principle rather as a curiosity.

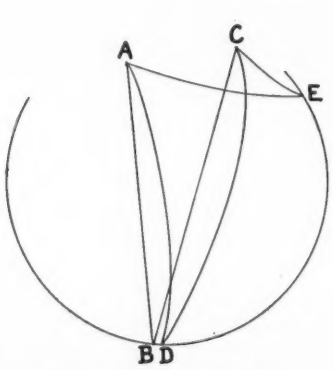


FIG. 1*a*. Reflection at a spherical mirror.

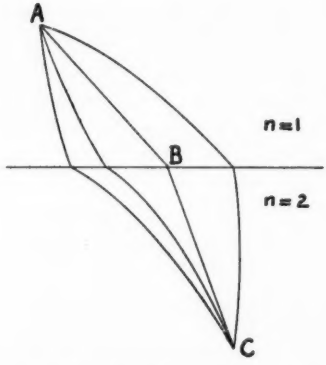


FIG. 1*b*. Refraction at a plane.

Let us see what (1) means. The integrand n is the *refractive index*. To the mathematician, it is a given function of position (and possibly of direction also); ds is an element of arc-length. The symbol δ tells us that a light ray is such that $\int n ds$, taken along the ray, has a stationary value when compared with the same integral taken along a neighboring path with the same end points. (The integral for any path is called its *optical length*.) Very often in optical problems the stationary value is actually a minimum, but this is not always the case. Fig. 1*a* shows reflection at a spherical mirror; the optical length of the ray ABC has a

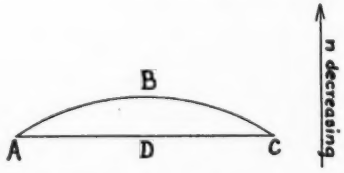


FIG. 2. Case where n varies continuously.

stationary optical length (in this case the same as geometrical length, if we take $n=1$) when compared with a neighboring curve ADC; but it is not a minimum (compare AEC). Fig. 1*b* shows refraction at a plane; here the ray ABC has a minimum optical length when compared with any other curve joining A and C.

In this latter case n is a discontinuous function, having one constant value below the horizontal line and another constant value above that line. Fig. 2 shows a ray ABC in the case where n varies continuously. Although the geometrical length of ABC is greater than that of the straight line ADC, its optical length is less. (Here we have the mathematics of the mirage.)

3. Electron optics. Optics is a very old subject—perhaps the oldest branch of mathematical physics. In recent times physicists have invented an instrument called the electron microscope. To the superficial mind there is only one feature in common between the electron microscope and the ordinary microscope—they both enable us to observe very small objects. But to the mathematical mind, intent only on the logical deduction of behavior on the basis of axioms, the two instruments are one and the same, or nearly so.

What are the axioms of the electron microscope? They are the axioms of electromagnetic theory, and we shall not trouble to write them down here. On the basis of these axioms, it is possible to prove that the path of an electron obeys a variational principle. This principle can be written precisely in the form (1). If we like (and we do like), we may wipe out the axioms of electromagnetic theory, and base electron optics on the axiom (1).

To the physicist, the integrand n in (1) has entirely different associations according as he is thinking of optics or electronic paths. To the mathematician it is just a given function. The only essential difference is that in electronic problems it is always a continuous (indeed, analytic) function, whereas in optics discontinuous functions most frequently occur. This distinction is insignificant in some arguments, important in others. For the detailed arguments developed later in this paper the distinction is important, as we find it necessary to assume analyticity. Consequently, the results are more significant for electron optics than for the classical optics.

4. The focal property. For three hundred years men have been grinding lenses and mirrors for a single purpose, and electron microscopes are constructed for that same purpose—the formation of images. It is therefore only fitting that a mathematician, sitting down to study the logical implications of the variational principle (1), should set before himself as an important problem the theory of the formation of images. Naturally, the first step is to get an idea (in mathematical terms) of what is meant by an image.

The easiest way to talk about these things is to use the language of physics without relaxing the clarity of mathematical thought. It is wise, however, not to insist on too great a precision of logic. Let us get the general picture, and leave the loose ends to another day.

In Fig. 3, O is a source of light and G a lump of glass. By means of a screen with a hole in it we cut off all the rays from O except a thin bundle. This bundle of rays strikes G and is refracted through it. Let us now take another screen and hold it in the path of the light, so that there is an illuminated patch on it. To make matters more precise, let us hold it always perpendicular to some assigned

ray of the bundle. What do we see as we move the screen to various positions between O and G and behind G?

Nothing exciting happens as we move the screen from O out towards G. The patch expands but remains of one shape. But on the other side of G the patch changes shape as the screen is moved. In each of two positions of the screen, the patch gets very thin.* In fact, if the initial angle of the bundle at O is regarded as infinitesimal, and small quantities of the second order are neglected, these thin patches are indistinguishable from straight lines. They are called *focal lines* (L_1, L_2).

There is so much confusion nowadays at to what is physics and what is mathematics, that is it well to pause for a moment and ask: Is this physics or mathematics? You can avoid the question by saying that it is mathematical physics, but let us rule that answer out of order. Then I would say without hesitation that it is mathematics, dressed up (if you like) in the language of physics.

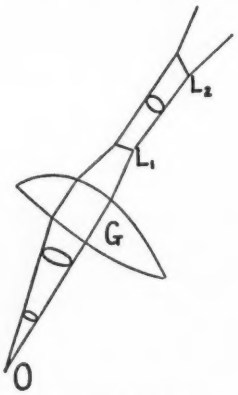


FIG. 3. Focal lines.

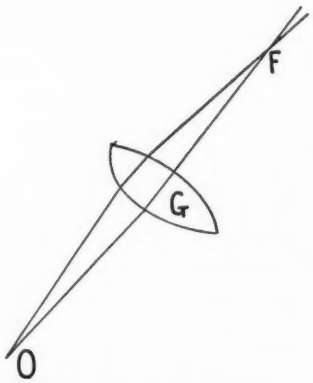


FIG. 4. The principal focus or image.

It is true that you can set up an experiment in a laboratory, and verify (with a certain amount of fuzziness) that the statements are physically true. But I have in mind a logical deduction of these statements from the equation (1) without the least trace of logical fuzziness. The deduction, of course, is not included here. It belongs, in the language of modern mathematics, to "differential-geometric aspects of the calculus of variations." It seems simpler to call it "geometrical optics."

But let us return to the question of the image. In Fig. 3 there is no image. But if we are lucky or clever in choosing the shape of the glass G, things may turn out as shown in Fig. 4. Here the two focal lines have disappeared. Instead there is a point F through which all the rays of the bundle very nearly pass. In

* An over-simplified statement. Sometimes the thin patches do not occur as stated, but exist only on an "imaginary" screen cutting the "imaginary" bundle of rays obtained by producing the final rays backward through G.

fact, they miss it only by a distance which is an infinitesimal of the second order. The point F is a *principal focus*, or an *image* of the *object* O.

Here we come to a climax, which should not be missed. The condition shown in Fig. 3 is *general*, that shown in Fig. 4 is *exceptional*. If, in the path of rays from O, you set up a lump of glass at random, the probability of getting an image is zero. An image occurs when the two roots of a quadratic equation coincide; that is an event of zero probability if the coefficients in the equation are chosen at random. (As a matter of fact, the coefficients are not completely random, since the roots are actually always real. But that does not affect the zero probability.)

Had Galileo stuffed chips of glass at random into his telescope tube, it is highly improbable that he would ever have seen the crescent of Venus. Clearly he had a design. What was its essential feature? How are we to change a probability zero into a probability unity?

The answer is simple: give the instrument symmetry of revolution, and point the axis of symmetry at (or near) the object. Practical physicists have known this so long that they forget that it is not quite obvious. When they came to construct electron microscopes, they used the same plan with success. Mathematical proofs have been given both for optical instruments and for the electron microscope. The most satisfying argument for the optical instrument is one based on Hamilton's angle-characteristic [2] (or eikonal [3]). A simpler argument has been given [4], and this may be used also for the electron microscope in the case where the paths of the electrons are straight lines before entering the instrument and after emerging from it.

When the electromagnetic field extends over the whole path of the electron (so that it is curved throughout), the arguments mentioned above cannot be used. The proofs given in this case [5] are rather involved, and I thought it would be interesting to discuss the problem using only the basic variational principle (1), without recourse to electromagnetic details until the last possible moment. When this plan is combined with the use of the complex variable, we get a very compact type of argument. Incidentally (as so often happens when a new method is tried) there is something in the way of a new result. It is found that symmetry less complete than symmetry of revolution is sufficient.

5. The differential equations and cases of symmetry. The variational equation (1) may be written

$$\delta \int m(x, y, z, x', y')dz = 0, \tag{2}$$

where x, y, z are rectangular Cartesian coordinates, the prime denotes d/dz, and

$$m(x, y, z, x', y') = n \cdot ds/dz. \tag{3}$$

The Euler equations of (2) read

$$\frac{d}{dz}\frac{\partial m}{\partial x'} - \frac{\partial m}{\partial x} = 0, \qquad \frac{d}{dz}\frac{\partial m}{\partial y'} - \frac{\partial m}{\partial y} = 0. \tag{4}$$

Assuming analyticity, we expand m:

$$
\begin{aligned}
m = {} & m_0 + m_1 x + m_2 y + m_3 x' + m_4 y \\
& + \tfrac{1}{2}(m_{11}x^2 + 2m_{12}xy + m_{22}y^2) \\
& + \tfrac{1}{2}(m_{33}x'^2 + 2m_{34}x'y' + m_{44}y'^2) \\
& + x(m_{13}x' + m_{14}y') + y(m_{23}x' + m_{24}y') + M,
\end{aligned} \tag{5}
$$

where M consists of terms of the third and higher orders. The coefficients are functions of z.

We come now to the kernel of the matter—the question of symmetry. This is best discussed in terms of "covering operations." A covering operation is a transformation of coordinates such that the specification or description of the instrument is the same whether we use the old or the new axes. We shall consider only covering operations which consist of rotations about the z-axis.

Let us rotate the axes Oxy into $O\bar{x}\bar{y}$ by turning them through an angle α. This gives the transformation

$$
x = \bar{x}\cos\alpha - \bar{y}\sin\alpha, \qquad y = \bar{x}\sin\alpha + \bar{y}\cos\alpha, \tag{6}
$$

with similar formulas for x', y'. As far as we are concerned mathematically, the "instrument" is essentially the function m. If m is an arbitrary function, it will be changed by the transformation (6). Rotation through the angle α is a covering operation if, and only if,

$$
m(x, y, z, x', y') = m(\bar{x}, \bar{y}, z, \bar{x}', \bar{y}'). \tag{7}
$$

The only possible types of rotational symmetry are

(i) symmetry of revolution (α arbitrary);

(ii) ν-gonal symmetry ($\alpha = 2\pi/\nu$, $\nu = 2, 3, \cdots$).

Fig. 5 illustrates some of these types of symmetry. As rotation through an arbitrary angle may be produced by an infinite succession of infinitesimal rotations, we may describe symmetry of revolution as ν-gonal symmetry with $\nu = \infty$.

On substituting (6) in (5) and comparing the form of the new expression with that of the old expression, we find that ν-gonal symmetry implies

$$
m_1 = m_2 = m_3 = m_4 = 0, \tag{8}
$$

with the following further relations, if $\nu > 2$:

$$
m_{11} = m_{22}, \quad m_{33} = m_{44}, \quad m_{13} = m_{24}, \quad m_{14} = -m_{23}, \quad m_{12} = m_{34} = 0. \tag{9}
$$

Changing the notation for the coefficients, we have, for ν-gonal symmetry with $\nu = 3, 4, \cdots$,

$$
\begin{aligned}
m = {} & p_0 + \tfrac{1}{2}p_1(x^2 + y^2) + \tfrac{1}{2}p_2(x'^2 + y'^2) \\
& + p_3(xx' + yy') + p_4(xy' - yx') + M;
\end{aligned} \tag{10}
$$

for diagonal symmetry ($\nu=2$), the expression (5) is simplified merely by the omission of the linear terms.

We note that the terms written explicitly in (10) are invariant under an arbitrary rotation. Hence, in any approximation which neglects M, *ν-gonal symmetry implies symmetry of revolution if $\nu>2$.*

Symmetry of revolution of course implies ν-gonal symmetry for all values of ν; for an instrument of revolution, we have (10) ,with further knowledge that M is a function of the invariants

$$x^2+y^2, \qquad x'^2+y'^2, \qquad xx'+yy', \qquad xy'-yx'.$$

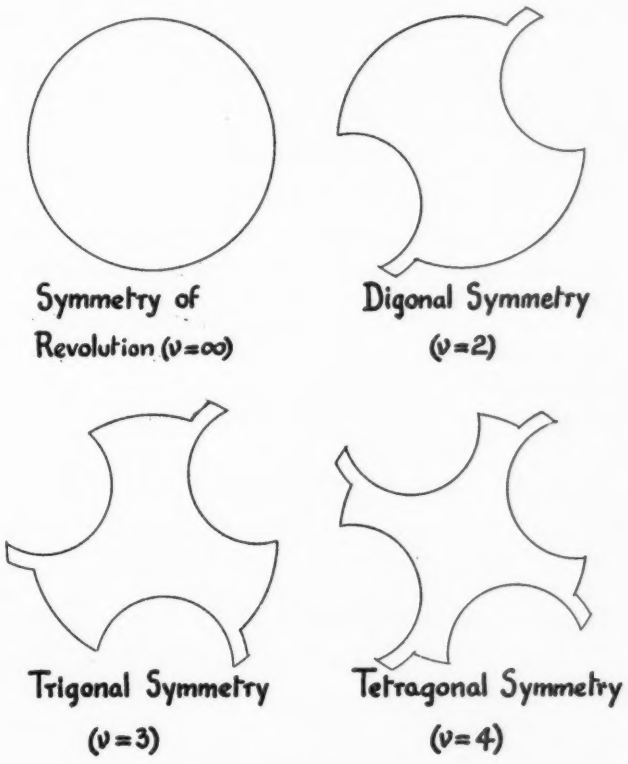


FIG. 5

6. Introduction of the complex vector and reduction of the equations to a single equation with real coefficients. Let us now confine our attention to the case of ν-gonal symmetry which $\nu>2$, so that m is in (10). Symmetry of revolution is included as a particular case. The Euler equations (4) read

$$\begin{aligned}&\frac{d}{dz}(p_2x' + p_3x - p_4y) - (p_1x + p_3x' + p_4y') = N_x,\\ &\frac{d}{dz}(p_2y' + p_3y + p_4x) - (p_1y + p_3y' - p_4x') = N_y,\end{aligned} \tag{11}$$

where

$$N_x = -\frac{d}{dz}\frac{\partial M}{\partial x'} + \frac{\partial M}{\partial x}, \qquad N_y = -\frac{d}{dz}\frac{\partial M}{\partial y'} + \frac{\partial M}{\partial y}. \tag{12}$$

In terms of the complex vector $\zeta = x + iy$, the two equations (11) may be written as the single equation

$$\zeta'' + P\zeta' + Q\zeta = T, \tag{13}$$

where

$$\begin{aligned}P &= (p_2' + 2ip_4)/p_2,\\ Q &= (p_3' + ip_4' - p_1)/p_2,\\ T &= (N_x + iN_y)/p_2.\end{aligned} \tag{14}$$

We shall not consider the singular case $p_2 = 0$; if $p_2 = 0$, the character of the differential equations (11) is altered drastically, since the second derivatives of x and y no longer appear on the left-hand side.

The fact that we can write (11) in the simple form (13) is itself remarkable. This could not be done but for the pattern of the coefficients in (11); this pattern arises from the symmetry of the instrument.

We now remove the second term from (13) by introducing the new complex vector

$$u = \zeta \exp\left(\frac{1}{2}\int P(z)dz\right). \tag{15}$$

We obtain

$$u'' + Ru = T \exp\left(\frac{1}{2}\int P(z)dz\right), \tag{16}$$

where

$$\begin{aligned}R &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2\\ &= [p_2(p_3' - p_1) - \tfrac{1}{2}p_2p_2'' + \tfrac{1}{4}p_2'^2 + p_4^2]/p_2^2.\end{aligned} \tag{17}$$

The reader should not miss the fact that something very remarkable has just occurred. From *complex* expressions for P and Q we have obtained a *real* R! It is Miracle No. 1 in the proof of the focal property of an electron microscope. It raises the probability of the existence of an image from 0 to $\frac{1}{2}$, as we shall see presently.

For paths adjacent to the axis of symmetry, ζ (or u) is a small quantity of the first order. The complex quantity T is a small quantity of the second order at least, for ν-gonal symmetry ($\nu>2$), and of the third order for symmetry of revolution. Unless we wish to study aberrations, we neglect T and write for (16)

$$u'' + Ru = 0. \tag{18}$$

This business of neglecting small quantities, apparently of higher order, without a thorough investigation, is (in the eyes of the pure mathematician) the greatest vice of the applied mathematician. Many a shining mathematical monument (the theory of small oscillations, for example) was built by ruthless builders who threw into the junkpile all inconvenient residues. "Forward!" they cried, "The problem must be linearized. We shall come back another day, and look into these residues." They seldom do. In our problem, the residue T contains the theory of aberrations. They are very important to the physicist, and have been investigated. But we shall stick to the linearized equation (18); it contains the secret of the focal property.

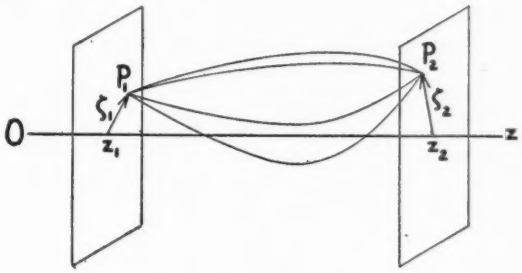


FIG. 6. Formation of an image.

If, however, we go back to (10) and delete M, *i.e.*, if we decide to deal with an instrument for which m is given by (10) with M deleted, then the equation (18) becomes exact, and it is no longer necessary to offer apologies. We have a choice between an inexact mathematical treatment of a general m and an exact treatment of an artificially simplified m. Whichever we choose, we arrive at (18).

Let us recall the general situation. Fig. 6 shows the instrument, Oz being the axis of symmetry. Extremals (or electron paths) start from the point P_1 with coordinates $z=z_1$, $\zeta=\zeta_1$. If the instrument has the focal property, then all extremals starting from P_1 in arbitrary directions pass through a common point P_2 (say, $z=z_2$, $\zeta=\zeta_2$). If these extremals are projected on the plane $z=0$, they will appear as in Fig. 7. These curves are solutions of (13) with T omitted, *i.e.*, of

$$\zeta'' + P\zeta' + Q\zeta = 0. \tag{19}$$

It seems too good to be true that the solutions of this equation should have the remarkable property of convergence shown in Fig. 7. And it is, in general, if P

and Q are arbitrary complex functions of the independent variable z. In our problem they are not arbitrary, but are so related that R is real.

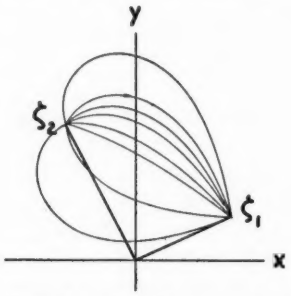


FIG. 7. Paths projected on normal plane.

Let us return to (18), remembering that though R is real, the solutions u in which we are interested are in general complex. Fig. 8 shows the complex u-plane, with the initial point P_1 ($u=u_1$). Let $f(z)$, $g(z)$ be independent *real* solutions of (18). The general complex solution is

$$u = \alpha f(z) + \beta g(z), \tag{20}$$

where α and β are arbitrary complex constants. The following particular solu-

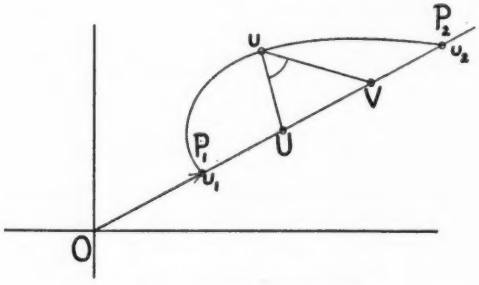


FIG. 8. The complex plane of u.

tions satisfy the initial condition $u=u_1$ for $z=z_1$

$$U = u_1 f(z)/f(z_1), \qquad V = u_1 g(z)/g(z_1). \tag{21}$$

As z varies, the points U, V move on the straight line OP_1.

Any other solution of (18) is of the form (20), and if this solution passes through $u=u_1$ for $z=z_1$, the constants satisfy

$$u_1 = \alpha f(z_1) + \beta g(z_1). \tag{22}$$

Then (20) may be written in either of the forms

$$u = u_1 f(z)/f(z_1) - \beta[f(z)g(z_1) - f(z_1)g(z)]/f(z_1) \tag{23a}$$

$$u = u_1 g(z)/g(z_1) + \alpha[f(z)g(z_1) - f(z_1)g(z)]/g(z_1). \tag{23b}$$

Thus

$$\frac{u - U}{u - V} = -\frac{\beta g(z_1)}{\alpha f(z_1)}. \tag{24}$$

This is independent of z, and so the angle between the vectors $u-U$ and $u-V$ remains constant as z changes. The ratio of the magnitudes of these vectors is also constant. From the constancy of the angle it follows that if U and V have a second coincidence (in addition to the initial coincidence at P_1), then u will be sucked into that coincidence. Such a coincidence (an image, in fact) occurs if the equation

$$f(z_2)/g(z_2) = f(z_1)/g(z_1) \tag{25}$$

has a real solution z_2, other than the trivial $z_2 = z_1$. For it is clear from (23) that such a value of z will make $u = U = V$. The position of the image is

$$z = z_2, \qquad u = u_2 = u_1 f(z_2)/f(z_1),$$

$$\zeta_2 = \zeta_1 \frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P(z)dz\right). \tag{26}$$

Since the equation (25) does not involve the initial lateral displacement ζ_1 (or u_1), it is clear that the existence or non-existence of an image does not depend on this lateral displacement. The focal property may be studied, if we like, by considering only the case $\zeta_1 = 0$, *i.e.*, the case where the object point is placed on the axis of symmetry. If the object point $z = z_1$, $\zeta = 0$ is imaged at $z = z_2$, $\zeta = 0$, then all the object points on the plane $z = z_1$ are imaged on the plane $z = z_2$. The transformation from one plane into the other is a uniform magnification accompanied by a rotation about the axis. If we prefer, we may speak of the "complex magnification"

$$\frac{\zeta_2}{\zeta_1} = \frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P(z)dz\right). \tag{27}$$

This may be split into a real magnification (positive or negative)

$$\frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P_1(z)dz\right), \tag{28}$$

and an angle of rotation

$$-\frac{1}{2}\int_{z_1}^{z_2} P_2(z)dz, \tag{29}$$

where $P = P_1 + iP_2$.

Before introducing the electromagnetic field, let us see where we have got to on the basis of the variational principle (1) alone. We have reduced the question of image formation to the discussion of the simple differential equation (18). The fact that R is real stands out as of prime importance. The next question of importance is this: Is R positive or negative? The variational principle alone does not restrict the sign of R. Without discussing the sign of R, however, we have brought the matter to a climax in the equation (25). It seems legitimate to say that there is probability $\frac{1}{2}$ that an arbitrary real equation should have a real root. Hence, symmetry alone raises the probability of an image from 0 to $\frac{1}{2}$.

As for symmetry, we have found that digonal symmetry is no help towards image formation, but that ν-gonal symmetry with $\nu>2$ is just as effective as symmetry of revolution. This means that in an electrostatic microscope circular holes are not essential. Trigonal or tetragonal holes might be used, and the same remark applies to the coils in a magnetostatic microscope. They might be triangular or square—but not rectangular. These suggestions are, however, more philosophical than practical, because it seems fairly certain that aberrations would be least in the case of symmetry of revolution.

7. The existence of real images in an electromagnetic instrument. If we attack the problem of the electron microscope directly through the equations of electromagnetism [6], we find the equation (19) with the following values for P and Q:

$$(30)\qquad P=\frac{1}{2}\frac{W'}{W}+i\Omega_0'\left(\frac{e}{2mW}\right)^{1/2},\qquad Q=\frac{1}{4}\frac{W''}{W}+\frac{1}{2}i\Omega_0'\left(\frac{e}{2mW}\right)^{1/2},$$

and hence by (17)

$$(31)\qquad R=\frac{3}{16}\frac{W'^2}{W^2}+\frac{1}{2}\Omega_0'^2\frac{e}{2mW}.$$

Here

e = electronic charge (negative),
m = electronic mass (positive),
$V_0(z)$ = axial electrostatic potential,
$\Omega_0(z)$ = axial magnetostatic potential,
$W(z)=C-V_0(z)<0$
C = constant of energy.

The radicals in (30) are positive.

The above formulas are most easily derived for the case of symmetry of revolution, but they hold also for ν-gonal symmetry with $\nu>2$.

There is an important difference between the general R of (17) and the R for the electromagnetic field as given by (31). The electromagnetic R is positive definite; it vanishes only if the field is absent. This is Miracle No. 2 in the theory of the electron microscope.

Let us take any point ζ_0, z_0, not on the axis of symmetry. Let us start a trajectory from this point in a direction making $u'=0$, u being given by (15). This does not, in general, imply $\zeta'=0$; thus the direction will not be parallel to the axis. Since R is positive, it is evident that the differential equation (18), under these initial conditions, defines a function $u(z)$ which vanishes at least once for $z<z_0$ and at least once for $z>z_0$. Let us call these points z_1 and z_2.

Now if we place an object on the axis at z_1, we get an image at z_2, since the vanishing of u implies the vanishing of ζ. The plane perpendicular to the axis at z_1 is imaged on the plane perpendicular to the axis at z_2, in accordance with (26). We have therefore this result: *Every statical electromagnetic system with an axis of ν-gonal symmetry (or symmetry of revolution) forms images (principal foci) of some objects, provided $\nu>2$.*

The reader familiar with the language of optics may ask whether the images of which we speak are "real" or "virtual." We say therefore that the virtual image plays no part in the theory as developed here. By "image" we mean "real image."

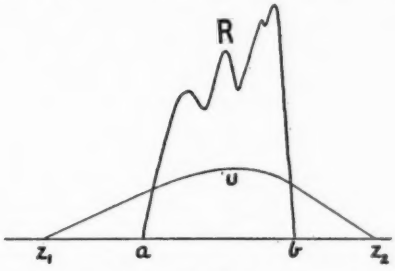


FIG. 9. Graphs of R and u.

We cannot assert that *every* object has an image, for in the above deduction we did not choose z_1 arbitrarily. The field bends in to a principal focus a set of trajectories which diverges weakly enough, or does not diverge at all, as when the object is at infinity. But the field may not be strong enough to bend in a strongly divergent set of trajectories. However, we can secure an image of any point if we are allowed to increase the strength of the field, *i.e.*, increase the positive coefficient R in (18). With the understanding that we are allowed to do this, we have reached our goal. The probability of securing an image has been raised from 0 through $\frac{1}{2}$ to 1.

I wish to thank my colleague, Professor T. Radó, for the interest he has taken in this work, and for his criticisms and suggestions. Among other results connected with the differential equation (18), he has pointed out one which concerns the case where $R(z)>0$ for $a<z<b$ and $R(z)=0$ for z outside this range. Then there is certainly an image if

$$z_1 < a - \left(\int_a^b R dz\right)^{-1}.$$

This case is illustrated in Fig. 9, which shows graphs of R and u.

References

1. G. D. Birkhoff, American Scientist, 31, 1943, 310.
2. J. L. Synge, Geometrical Optics, Cambridge, 1937, 68.
3. C. Carathéodory, Geometrische Optik, Berlin, 1937, 84.
4. E. G. Ramberg and G. A. Morton, Journal of Applied Physics, 10, 1939, 465–478.
5. J. Picht, Einführung in die Theorie der Elektronenoptik, Leipzig, 1939, 15, 70.
6. J. L. Synge and B. A. Griffith, Principles of Mechanics, New York, 1942, 383, 385.

INTERESTING THE ENGINEERING STUDENT*

W. L. AYRES, Purdue University

1. Introduction. I hope no one will expect a complete solution to the problem mentioned in the title. While I have taught engineering students in two different institutions over a considerable period, this is only a partially solved problem as yet to me. From time to time I have experimented with different methods to attract their interest to mathematics. This paper might be termed a preliminary report, for my experiences are continuing and I expect to have additional ideas concerning this problem in years to come.

2. The problem. Although we speak of him in the singular, the engineering student varies in many characteristics and qualities. Nowhere is this variation more striking than in the mathematics classroom. At one end of the scale we have the students who were particularly excellent in mathematics in high school and perhaps entered the study of engineering because, so far as they knew, it was the only field in which mathematics could be used to earn a living. At the other end of the scale there are the students who chose the profession of engineering because they were particularly skilled with hand tools or wished an occupation with outdoor work. This may seem facetious but is actually the reason why many students enter our engineering colleges. This is most unfortunate as such students have little aptitude for the mathematics and other theory encountered in the engineering curriculum and belong really in trade schools.

There is no problem of interest with the student of the first type. In fact our problem, if there is one, is to see that he does not become too interested in mathematics. To a mathematician this is probably heresy, but from an engineering viewpoint there is certainly a possibility that the student may become too deeply interested in mathematical theory to the detriment of its engineering applications. In such a case, the student should be encouraged to specialize in engineering design where full use can be made of his mathematical interest or to change from engineering into the field of mathematics itself. Some of these students are not in this field only because the proper vocational counseling was not available to them during their high school days.

* Delivered June 20, 1943, before the Society for the Promotion of Engineering Education at Chicago.

Probably a student at the other end of the scale cannot be interested in mathematics at all, and needs some advice which will lead him from the engineering profession into a trade school or other mechanical work where his ability with tools is a real advantage.

In between these two extremes lie many different layers of ability and types of personalities, and the problem of interest in mathematics is concerned primarily with this intermediate and largest group. Too many of these students find mathematics dull and exacting, and breathe a sigh of relief when their required mathematics courses are finished. Then, in the language of the student, he can close his books and forget mathematics. This is precisely the point of view we must fight if our mathematics teaching is to be successful. A student who has this feeling will never develop into an engineer of the highest quality, and it is to the interest of both the engineering profession and ourselves as mathematics teachers to see that this attitude does not develop. Unfortunately these two groups have not always cooperated in putting mathematics in the proper perspective for students of engineering.

3. Faults of the mathematics teacher. Too often the mathematician has been at fault in this respect. He has too frequently looked on the liberal arts students who elect to major in mathematics as the group deserving his best efforts, and on his engineering class as a part of his job, necessary to earning his salary, but not requiring his full energy in teaching the power and beauty of mathematics.

I once heard a young mathematics instructor refer to his engineering class as "my dummies." Needless to say, this attitude did not result in proper teaching. Very shortly his performance became so poor the engineering class was taken from him and he soon left the field of teaching altogether. While this is an extreme case, it has been my observation that many professional mathematicians attempt to avoid teaching engineering classes. From a purely professional point of view this is most unfortunate as these classes contain some of the finest products of our high schools. As mentioned earlier, some of these students do not belong in engineering at all, but in the field of mathematics. A considerable number of fine graduate students in mathematics were originally in engineering classes, and one could easily name several professional mathematicians who began as engineering students and in some cases hold engineering degrees. Thus the mathematician looking for students to be led along the path of mathematics should realize that in his engineering classes he has a fertile source of students, perhaps the most fertile source of all.

Moreover, in the engineering classes there is a certain reality of interest in mathematics not found by the liberal arts student who takes a course or two as a cultural elective. Here are students who are to use their mathematics in the profession of engineering. This should be more of a stimulus to the teacher than the girl who elects a course in the calculus with no intention whatever of using the calculus at any time. For this reason alone it seems that the teacher of mathematics truly interested in his subject would find the engineering classes exciting.

4. Faults of the professional engineer. On the other hand, teachers of technical courses as well as the practising engineer have sinned frequently by giving students the impression that their mathematics is not important.

Very often the engineer becomes an executive or enters the sales division, and forgets that he is not really practising his profession at all. In these abortive lines naturally he has little need for his mathematics, and often he is quite vocal in his scorn for mathematics. He continues to think of himself as an engineer and overlooks the fact that he himself, rather than the engineering profession, has ceased to use mathematics. Similarly a doctor who became a drug manufacturer might advocate that courses in anatomy in the medical curriculum be replaced by cost accounting.

Sometimes we find engineering professors who fear and mistrust mathematical methods. Usually this is caused by their lack of knowledge of the technique and powers of mathematical methods. This leads to curious incidents. I know one engineering professor who told his class that all the engineer needs of mathematics is arithmetic and perhaps a little trigonometry. You can imagine the profound impression this produced in the minds of these young and immature young men, and the difficulties we had in the next few weeks combating this impression. The sequel to this story is that this same professor came to the mathematics office a few weeks later for some assistance in reading an article in his own field. A knowledge of Bessel functions was required for its reading and this was unknown territory to him. I am delighted to say that he was reminded of his earlier remarks and it was suggested that he try arithmetic and a little trigonometry on the article.

Mathematics teachers are sometimes embarrassed by questions from junior and senior students who inquire why Professor X does not use mathematics in teaching the theory of his subject. One hates to face such questions for usually the facts are that Professor X is not familiar with modern mathematical methods and is applying an earlier rule or some method he has learned which does not involve calculus and beyond. Frequently mechanics is taught so as to avoid calculus wherever possible. This is deplorable on two counts. In the first place the mathematical methods are more elegant and powerful. In addition it is imperative that the students continue to apply their mathematical ideas so that these ideas and methods become firmly impressed on their minds. If the teacher of technical subjects is not to use these mathematical tools, perhaps the mathematical requirements in engineering education should be abolished and engineering taught as a trade profession.

Having relieved ourselves of this venom against both our own colleagues and the professional engineer, we may turn to one or two constructive ideas on the question of interesting engineering students in mathematics.

5. Engineering textbooks. This places me in a most embarrassing position for I wish to advocate the writing of some new textbooks for the mathematics taught in the engineering college. All my life I have taken the stand that there

were plenty of textbooks already available, in fact, that the supply was more than ample. With few exceptions, however, all our textbooks have been written from a commercial viewpoint. That is, each attempts to appeal to as diverse a group of college classes as possible in order to produce the largest possible sale. This results in a textbook which fits all needs to an epsilon approximation but sometimes the epsilon seems disturbingly large. The number of engineering students in this country is sufficient to justify a textbook being written specifically for them and I believe such a book or series of books well written would contribute much in this problem of interest.

In the problems which were collected and edited by a committee of the Society for the Promotion of Engineering Education under the chairmanship of J. W. Cell, and which have just been published in book form by the McGraw-Hill Book Company, a real start in this direction has been made. However, this is only a beginning and real satisfaction will not be achieved until problems of this type and similar material from the various fields of engineering find their way into the mathematics classroom, not as supplementary exercises, but as a regular part of the courses. Until we integrate them into the courses and into the textbooks used, the average student about whom this paper worries will not look on them as important.

In addition to our familiar x, y, z letters, these new textbooks should use the symbols in common use in the various divisions of engineering literature. In this way the student becomes accustomed to seeing his mathematics in various letters. While this seems trivial, it is a real hurdle to the average student. My friends in the chemistry faculty tell me the student who can solve a quadratic equation with ease when the variable is x and the location is the mathematics building, finds the greatest difficulty when quadratic equations confront him in a chemistry course using chemical letters. I see no reason why the student should be given twenty problems of the type $4x^2-7x+3=0$ when we can introduce into this set of problems examples of quadratic equations which occur in engineering subjects and he can see the same method applied to these problems.

The textbook should give the student some training in problems where coefficients have one or two decimal places in place of the more elegant whole numbers of mathematics. Most of his equations in technical work will have such approximate numbers and unless he has also seen them in the mathematics classroom he may not make the desired transfer of his training. In exponents, too, the students should meet a few numbers with decimal figures.

These proposed textbooks should show the student a few applications of complex numbers so as to remove the ghostly character of imaginary numbers. It should explain that the electrical engineer uses the letter j instead of i and why he does so. In this way the student continually feels the reality of his mathematics and it is tied closer to the profession which lies close to his heart.

It is unbelieveable that one can teach the three types of conics without showing the class where such curves occur in the world about him. Yet there are

several textbooks where these applications are not even mentioned. Here, as elsewhere, the subject of mathematics has been divorced from reality and the engineering student loses thereby.

In the calculus the situation is not so serious, for all textbooks contain many applications. The student applies his techniques to maxima and minima problems and to find moments, centers of gravity, volumes bounded by surfaces, *etc.* Here the supply of applications, while perhaps not perfect, is somewhat adequate.

It is in trigonometry, algebra, and analytic geometry that the student is not shown enough of the applications. Many students study trigonometry and see no purpose to it at all except in surveying and other measuring of plane areas. With the richness of applications of this subject, there is no reason why the student should not be acquainted with many other fields where this material finds use. In a trigonometry class recently, one of our instructors was asked about the usefulness of angles greater than 180° and he put the question back to the class. Just one member of the class knew of an application where it was necessary to use angles of more than 180°. Many of our textbooks teach these angles with no mention of their use. So the student, thinking entirely of his trigonometry in terms of triangles, sees no reason for studying them except for the need of a passing grade.

Throughout his entire training in mathematics, the application of the subject to engineering should be emphasized over and over again. This does not mean we are to teach engineering in our mathematics classrooms, but the problems and formulas of engineering should be introduced in the textbooks whereever it can be done without discussing technical matters. This is not done at present, partly because suitable applications are difficult to find and present to students with no technical background, and partly because the mathematics teacher is not sufficiently acquainted with the applications of his subject.

6. Algebra and its technique. If there is one subject above all in mathematics which the student finds dull and divorced entirely from his reality, that subject is algebra. Every mathematician knows that algebraic technique is the foundation of all mathematics and yet the student sees only an endless series of arbitrary rules and does not see the value of the subject. A student is forced to learn to handle exponents, to solve quadratic equations, to solve higher degree equations, and all the other techniques of elementary algebra. He sees no connection with bridges, dynamos or internal combustion engines. As a result I find this subject the most difficult in the matter of student interest.

Some of the proposals of the previous section will assist in removing this difficulty, but I believe it would improve matters also to combine algebra with analytic geometry. For example, determinants and systems of linear equations may be studied along with the subject of straight lines in analytic geometry. And the methods of handling quadratic equations would accompany the conics of analytic geometry. In this way we may motivate some of the study of algebraic technique.

In general this matter of technique is one of the most difficult problems the mathematics teacher faces. In mathematics we attempt to do two things simultaneously: to impart new ideas, and to teach the students the technique of handling these new methods. It is useless to teach the idea without drill on technique, and yet the student finds the technique laborious and often dull. A too heavy emphasis on technique is sure to produce boredom on the part of the student. In the long run more can be accomplished by attempting to hold the interest of the student even if this means slightly less drill and perfection in technique. The student who finds his mathematics exciting will perfect his technique satisfactorily, whereas the student who has technique drilled into him against his desire will hate the entire subject and never learn it at all.

7. Conclusion. Finally this matter of interesting the engineering student reduces to a problem in salesmanship. We must sell our students the idea that mathematics is interesting, that it is important, and that it will be useful to them later. It is necessary that the mathematics teacher and the teacher of technical subjects cooperate in this undertaking. It is to their interest as much as ours to see that the student learns his mathematics and approaches his technical subjects with clear and definite mathematical ideas. Only when both groups join in facing this problem will we make real progress toward our goal.

POLYGONS OF GREATEST AREA INSCRIBED IN AN ELLIPSE

W. V. PARKER and J. E. PRYOR, Louisiana State University

1. Introduction. In many books on the calculus the problem is given to determine the isosceles triangle of greatest area or the rectangle of greatest area which can be inscribed in a given ellipse. The purpose of this paper is to discuss polygons of greatest area inscribed in an ellipse and polygons of least area circumscribed about an ellipse. The term polygon as used throughout this discussion is to mean a simple convex polygon. A polygon of greatest area will be referred to as a maximum polygon and a polygon of least area will be referred to as a minimum polygon. Two lemmas will be used as a basis for the discussion.

2. Maximum polygon inscribed in a circle. The existence of a maximum polygon is assumed. However, this may be proved by elementary analytical methods. The coordinates of the vertices of a polygon of n sides inscribed in a circle with unit radius and center at origin are designated by $(\cos\theta_i, \sin\theta_i), \theta_i < \theta_{i+1}$ and $\theta_{n+1} = 2\pi + \theta_1$. Let $u_i = \theta_{i+1} - \theta_i$ and the area of the polygon is

$$A = \tfrac{1}{2}\sum_{i=1}^{n} \sin u_i = \tfrac{1}{2}\sum_{i=1}^{n-1} \sin u_i - \tfrac{1}{2}\sin\sum_{i=1}^{n-1} u_i.$$

Hence, $\partial A/\partial u_j = \tfrac{1}{2}\cos u_j - \tfrac{1}{2}\cos\sum_{i=1}^{n-1} u_i = \tfrac{1}{2}\cos u_j - \tfrac{1}{2}\cos u_n$, $(j = 1, 2, \cdots, n-1)$, and a necessary condition that A be maximum is that $u_j = u_n = 2\pi/n$. This is also

a sufficient condition since $\Delta_p = |a_{jk}| = (-1)^p(\frac{1}{2})^p(p+1)\sin^p 2\pi/n = (-1)^p N_p$ $(N_p > 0)$ where $a_{jk} = \partial^2 A/\partial u_j \partial u_k$ and $u_j = u_n = 2\pi/n$, $(j, k = 1, 2, \cdots, p)$.

LEMMA 1. *The maximum polygon of n sides which can be inscribed in a circle is a regular polygon.*

Let $P_1, P_2, \cdots, P_n$ denote the consecutive vertices of a polygon inscribed in a circle. If any two adjacent sides, say P_1P_2 and P_2P_3, of this polygon are of unequal lengths then a larger inscribed polygon of n sides may be formed. For if all vertices except P_2 are kept fixed and P_2 is moved to the point of its arc which is the greatest distance from the chord P_1P_3 the area of the triangle $P_1P_2P_3$ will be increased and since the remainder of the polygon remains unchanged the area of the polygon is increased. Hence if any inscribed polygon of n sides is not regular, it is always possible to get an inscribed polygon of n sides which has a greater area. It is generally not possible to begin with a polygon which has two or more sides of unequal lengths and arrive at a regular polygon by repeating this process a finite number of times. However this process gives a bounded increasing sequence of areas whose limit must be the area of a regular polygon.

3. Minimum polygon circumscribed about a circle. The existence of a minimum polygon is also assumed. However, this may be proved analytically as in Lemma 1. Denote the points of tangency of the sides by $(\cos\theta_i, \sin\theta_i)$. The area is

$$A = \sum_{i=1}^{n} \tan\tfrac{1}{2}u_i = \sum_{i=1}^{n-1} \tan\tfrac{1}{2}u_i - \tan\tfrac{1}{2}\sum_{i=1}^{n-1} u_i.$$

A necessary and sufficient condition that A be minimum is that $u_i = 2\pi/n$ $i = 1, 2, \cdots, n$.

LEMMA 2. *The minimum polygon of n sides which can be circumscribed about a circle is a regular polygon.*

If in a circumscribed polygon of n sides the interior angles at two adjacent vertices are unequal, it is possible, by moving the side joining these vertices and leaving the other $n-1$ sides fixed in position, to get a circumscribed polygon of n sides which has a smaller area. By argument similar to that used above it follows that the minimum circumscribed polygon must have all interior angles equal and consequently is a regular polygon.

4. Polygon inscribed in an ellipse. There is a regular inscribed polygon of n sides with a vertex at any given point of the circle and these all have the same area. Hence by projection* there is a maximum inscribed polygon of n sides for an ellipse with a vertex at any point of the ellipse and these all have the same area. If the semi-axes of the ellipse are a and b we have the following result.

* See Fine, H. B. and Thompson, H. D., Coordinate Geometry, 1909, p. 92, Art. 118. The transformation $x' = x$, $y' = by/a$ carries the circle $x^2 + y^2 = a^2$ into the ellipse $x'^2/a^2 + y'^2/b^2 = 1$. An area A in the circle is projected into an area b/aA in the ellipse.

THEOREM 1. *The area of the maximum polygon of n sides which can be inscribed in an ellipse is $(n/2)ab \sin 2\pi/n$ and the difference between the eccentric angles for every pair of adjacent vertices of such a polygon is $2\pi/n$.*

This area is the geometric mean between the area of a regular polygon of n sides in the major circle and the area of a regular polygon of n sides in the minor circle. Also by projection we have the following result.

THEOREM 2. *The area of the minimum polygon of n sides which can be circumscribed about an ellipse is $nab \tan \pi/n$ and the difference between the eccentric angles of the points of tangency of every pair of adjacent sides of such a polygon is $2\pi/n$.*

Since every regular polygon of n sides circumscribed about a circle has its vertices on another circle it follows that the minimum polygons of n sides circumscribed about an ellipse are also maximum polygons inscribed in another ellipse. It follows that the minimum polygons of n sides circumscribed about the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

are the maximum polygons of n sides inscribed in the ellipse

$$x^2/a^2 + y^2/b^2 = \sec^2 \frac{\pi}{n}.$$

Also the maximum polygons of n sides inscribed in the ellipse (1) are the minimum polygons of n sides circumscribed about

$$x^2/a^2 + y^2/b^2 = \cos^2 \frac{\pi}{n}.$$

The transformation

$$\begin{cases} x' = x \cos \alpha + y \dfrac{a}{b} \sin \alpha \\ y' = -x \dfrac{b}{a} \sin \alpha + y \cos \alpha \end{cases} \tag{2}$$

will carry any polygon inscribed in the ellipse (1) into another inscribed polygon of the same area. However if these polygons are maximum inscribed polygons for the ellipse there are other invariants besides the area which are of interest.

5. Invariant properties of maximum and minimum polygons. One of the results is expressed in the following way.

THEOREM 3. *The sum of the squares of the lengths of the sides of every maximum polygon of n sides inscribed in the ellipse* (1) *is $2n(a^2+b^2) \sin^2 \pi/n$, and the sum of the squares of the lengths of the sides of every minimum polygon of n sides circumscribed about the ellipse* (1) *is $2n(a^2+b^2) \tan^2 \pi/n$.*

The second of these relations follows immediately from the first in view of the above discussion, so it is sufficient to prove the first relation.

Let $(a \cos \theta_i, b \sin \theta_i)$, $i=1, 2, \cdots, n$, be the vertices of a maximum polygon of n sides inscribed in the ellipse. Since $\theta_{i+1}-\theta_i=2\pi/n$, $\theta_i=\theta_1+(i-1)2\pi/n$. Define $\theta_{n+1}=\theta_1+2\pi$. Let S denote the sum of the squares of the lengths of the sides of the polygon. Then

$$S = \sum_{i=1}^{n} \{a^2(\cos \theta_{i+1} - \cos \theta_i)^2 + b^2(\sin \theta_{i+1} - \sin \theta_i)^2\}$$

$$= 4 \sin^2 \frac{\pi}{n} \left\{a^2 \sum_{i=1}^{n} \sin^2 \left(\theta_i + \frac{\pi}{n}\right) + b^2 \sum_{i=1}^{n} \cos^2 \left(\theta_i + \frac{\pi}{n}\right)\right\}.$$

Since

$$\sum_{i=1}^{n} \cos^2 \{\alpha + (i-1)\beta\} = \frac{n}{2} + \frac{1}{2} \cos \{2\alpha + (n-1)\beta\} \sin n\beta \csc \beta,$$

it follows that

$$\sum_{i=1}^{n} \cos^2 \left(\theta_i + \frac{\pi}{n}\right) = \sum_{i=1}^{n} \cos^2 \left\{\left(\theta_1 + \frac{\pi}{n}\right) + (i-1)\frac{2\pi}{n}\right\} = \frac{n}{2},$$

and

$$\sum_{i=1}^{n} \sin^2 \left(\theta_i + \frac{\pi}{n}\right) = \sum_{i=1}^{n} \left\{1 - \cos^2 \left(\theta_i + \frac{\pi}{n}\right)\right\}$$

$$= n - \sum_{i=1}^{n} \cos^2 \left(\theta_i + \frac{\pi}{n}\right) = \frac{n}{2}.$$

Therefore

$$S = 2n(a^2 + b^2) \sin^2 \frac{\pi}{n}.$$

It is noted here that the sum of the squares of the lengths of the sides of the maximum polygon of n sides inscribed in an ellipse is the arithmetic mean between the sum of the squares of the lengths of the sides of a regular polygon of n sides inscribed in the major circle and the sum of the squares of the lengths of the sides of a regular polygon of n sides inscribed in the minor circle.

If r_i $(i=1, 2, \cdots, n)$ are the distances from the center of the ellipse (1) to the vertices of a maximum inscribed polygon then

$$R = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} \{a^2 \cos^2 \theta_i + b^2 \sin^2 \theta_i\}$$

$$= a^2 \sum_{i=1}^{n} \cos^2 \left\{\theta_1 + (i-1)\frac{2\pi}{n}\right\} + b^2 \sum_{i=1}^{n} \sin^2 \left\{\theta_1 + (i-1)\frac{2\pi}{n}\right\}$$

$$= \frac{n}{2}(a^2 + b^2).$$

It follows that if t_i $(i=1, 2, \cdots, n)$ are the distances from the center of ellipse (1) to the points where the sides of a maximum polygon of n sides inscribed in (1) are tangent to the ellipse

$$x^2/a^2 + y^2/b^2 = \cos^2 \frac{\pi}{n}$$

then

$$T = \sum_{i=1}^{n} t_i^2 = \frac{n}{2}(a^2 + b^2) \cos^2 \frac{\pi}{n}.$$

If p_i $(i=1, 2, \cdots, n)$ are the perpendicular distances from the center of the ellipse (1) to the sides of a maximum inscribed polygon of n sides then

$$\frac{1}{P} = \sum_{i=1}^{n} \frac{1}{p_i^2} = \sum_{i=1}^{n} \frac{a^2 \sin^2 \dfrac{\theta_{i+1} + \theta_i}{2} + b^2 \cos^2 \dfrac{\theta_{i+1} + \theta_i}{2}}{a^2 b^2 \cos^2 \dfrac{\pi}{n}}$$

$$= \frac{1}{a^2 b^2 \cos^2 \dfrac{\pi}{n}} \sum_{i=1}^{n} \left\{ a^2 \sin^2 \left(\theta_i + \frac{\pi}{n}\right) + b^2 \cos^2 \left(\theta_i + \frac{\pi}{n}\right) \right\}$$

$$= \frac{n(a^2 + b^2)}{2a^2 b^2 \cos^2 \dfrac{\pi}{n}}.$$

Similar invariants for the minimum polygons of n sides circumscribed about ellipse (1) may be obtained from these.

It might be pointed out that if a polygon inscribed in ellipse (1) is not maximum then the sum of the squares of the lengths of the sides does not remain invariant under transformation (2). Transformation (2) leaves every ellipse of the family $x^2/a^2+y^2/b^2=k$ unchanged. A maximum polygon of n sides inscribed in an ellipse of this family is transformed into itself by (2) if α is an integral multiple of $2\pi/n$. A necessary and sufficient condition that a polygon of n sides inscribed in an ellipse of this family be a maximum inscribed polygon of n sides is that there exist a transformation (2) different from the identity which transforms the polygon into itself.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

ANNUAL REPORTS AND TOPICS FOR CLUB TALKS

Officers in charge of Mathematics Clubs are requested to send in their annual reports for 1943–44 promptly at the close of the academic year. The form adopted for printing reports this last year is different from that used in the past, and should be followed as closely as possible. Titles and speakers are displayed on separate lines. Officers are listed at the end.

The editor would be glad to receive articles for publication in this department which either have been or might be used as Mathematics Club talks on an undergraduate level.

HISTORICAL NOTE

The approximation for the smallest angle in a right triangle, $A \cong 172°a/(b+2c)$, which appeared in a recent issue of this MONTHLY (Vol. 50, no. 10, p. 622, Dec. 1943), was recognized by W. A. Hurwitz as having previously appeared in this MONTHLY in 1920 (Vol. 27, no. 8, p. 368) in an article by R. A. Johnson. Professor Hurwitz, then editor of Questions and Notes, gave a bibliography (p. 366) in which the formula is traced to Nicolaus Cusanus in 1514, and the proof to Ozanam in 1699. Despite its antiquity, however, this formula seems to be not well-known by mathematicians today.

CLUB REPORTS 1942-43

Pi Mu Epsilon, Montana State University

Talks included on the program for 1942–43 were:
Partially ordered systems, by Roy Dubisch.
Cryptography, by Walter Hook.
A geometrical presentation of complex numbers, by Francis Pott.
Lebesgue integration, by Eldon Whitesitt.

The annual banquet was held in March, and was in honor of Professor E. F. A. Carey who retires this year. In the fall the annual *Pi Mu Epsilon* Mathematical Prize Competition for freshmen was held, and prizes of twenty-five, fifteen, and ten dollars were awarded respectively to Dick Burgess, Jack Groene, and Mary Farrington. At the June Commencement the *Pi Mu Epsilon* prize for excellence in mathematics was awarded to Francis Pott, and the *Pi Mu Epsilon* prize for excellence in physics was awarded to Mark Jacobson. The award this year consisted of a copy of *Willard Gibbs* by Rukeyser. Officers for the year were: Director, Francis Pott: Treasurer, Mel-Iden Pollock.

Alfred University Mathematics Club

How to count, was the topic presented by Professor Lloyd Lowenstein at the first meeting of the club, covering material on the correspondence of numbers and sets, as well as a discussion of the transfinite. Other programs were:

The miniature world, a finite geometry, by W. B. Cottrell, President.

Equations and their roots, by Mrs. T. H. Dunkelberger.

Topology, by Lewis Butler.

Paradox lost and paradox regained, a program presented jointly by Jeanette Torrey and Robert Burgess, two seniors in the Mathematics Department.

The yearly project of the club was the binding of several years' collections of mathematical publications which were accumulating in the library. The annual Blue Key Mathematics Award was presented to President W. B. Cottrell on Moving-up Day. The activities of the club were conducted by its officers for the past year, who were: President, W. B. Cottrell; Secretary, Jeanette Torrey; Treasurer, Richard Rulon. New officers elected at the final meeting were: President, Lewis Butler; Secretary, Frank Olson; Treasurer, Richard Rulon. Professor Lowenstein is the Club's faculty adviser.

M. I. T. Mathematical Society

The M. I. T. Mathematical Society heard seven speakers during the past academic year, on the following subjects:

Relaxation oscillations, by Professor Norman Levinson.

Functions of several complex variables, by Professor W. T. Martin.

Vectors and tensors, by Professor D. J. Struik.

Continued fractions, by W. S. Loud.

Numbers, by R. Baumberger.

Motors, by D. L. Thomsen.

Permissible functions in physics, by E. J. Zadina.

The Society sponsored a contest for freshmen and sophomores in May, 1942. The winners were G. Raisbeck and L. C. Biedenharn. Officers for 1942–43 were: President, W. S. Loud; Program Manager, R. Baumberger.

Pi Mu Epsilon, Oklahoma A. and M. College

The mathematical activities at the Oklahoma A. and M. College have been under the sponsorship of the chapter of *Pi Mu Epsilon*. The fraternity held two open meetings, besides several smaller meetings to discuss pledging, initiation, *etc.* Topics discussed at the open meetings were:

The essence of mathematics, by Professor J. H. Zant, Acting Head of the Department of Mathematics.

The ratios of similitude, by Professor B. F. Behrendt, of the Department of Civil Engineering, discussed from the standpoint of the engineer.

Eighteen new members were initiated during the year, though most of them have already been called into the armed services. The officers are: President, Lawrence Hanna; Secretary, Joe Oursler; Faculty Sponsor, Professor J. H. Zant.

The Archimedeans, Winthrop College

Among the interesting programs presented during 1942–43 were:

A demonstration of the slide rule, by Jessie Cockfield.

Pantomimes portraying great mathematicians and their contributions.

Aspects of Chinese mathematics.

On one occasion *The Archimedeans* and the National Council of Teachers of Mathematics entertained a group of air cadets, then stationed at Winthrop College, at an informal gathering. Officers for 1943–44 are: President, Jessie Cockfield; Vice-President, Emily Jean Adams; Secretary, Ann Major; Treasurer, Sarah Ellen Leslie; Social Chairman, Eloise Dempsey; Faculty Adviser, Mrs. Blanche Badger.

Junior Mathematical Club, University of Chicago

During the year the following papers were presented by students:

Repeating phenomena in the decimal representation of numbers, by Daniel Zelinsky.

Infinite series, by C. F. Brumfiel.

Invariants of quadratic differential forms, by Robert Kates.

Conjugate nets in asymptotic parameters, presented by Janet MacDonald to the *Senior Mathematical Club*. As a prize for the best student paper in content and presentation, Miss MacDonald was awarded a copy of Bell's *Development of Mathematics*.

In addition to the student papers listed above, the following talks were presented to the Club:

Mathematics and meteorology, by Professor W. T. Reid.

Alignment charts and their applications to meteorology, by Albert Cahn, Institute of Meteorology, University of Chicago.

Mathematics for electronics, by Professor M. R. Hestenes.

Singular points on higher plane curves, by Professor M. I. Logsdon.

Concerning the validity of the multiplication law of radicals, by Dr. Luise Lange, Wilson Junior College.

Alignment charts, by Professor L. R. Ford, Illinois Institute of Technology.

Permutation groups in cryptanalysis, by Professor A. A. Albert.

The generalized harmonic oscillator, by Stuart Lloyd, Department of Physics, University of Chicago.

Applications of mathematical logic to neuron networks, by Walter Pitts, Department of Biophysics, University of Chicago.

Almost-periodic functions, by Dr. Y. K. Wong.

In addition to the teas given before the talks, the club sponsored a party in the Autumn Quarter and a picnic in the Spring Quarter. The officers for the year 1942–43 were: President, Anne Lewis; Treasurer, Daniel Zelinsky; Social Chairman, Janet MacDonald; Committee, Robb Jacoby and Hyman Zimmerberg. The officers elected for 1943–44 are: President, Daniel Zelinsky; Treasurer, Hyman Zimmerberg; Social Chairman, Mary Toft; Committee, Robb Jacoby, Robert Kates, Charles Nichols.

Kappa Mu Epsilon, Chicago Teachers College

Illinois Gamma Chapter of *Kappa Mu Epsilon* has now completed its first year of fraternal activity. Heading the list of program meetings were the following discussions:

The struggle for existence, by Dr. Ralph Mansfield of the College faculty—a talk on developing a theoretical and mathematical formula for the struggle for existence of the mosquito.

Celestial navigation, by Naval Lt. F. C. Chunn.

Old and new ways of solving equations, by Dr. L. R. Ford, chairman of the Mathematics Department of the Illinois Institute of Technology.

How mathematics can be used to obtain vital information for an educational testing program, by Dr. W. C. Krathwohl, head of the Department of Tests and Measurements, Illinois Institute of Technology.

In addition to their charter members, the group initiated eighteen into their organization during the past year. Officers for the year were: President Archimedes, Robert Healy; Vice-President Euler, John Wiegand; Secretary Galileo, Arletta Mae Loomis; Treasurer Kepler, Rita Cooney; Secretary Descartes and Faculty Sponsor, Joseph Urbancek.

The Cooper Union Mathematics Club

Activities of the Mathematics Club of the Cooper Union School of Engineering for the academic year 1942–43, ending February 1943, were rather restricted because of the continuing accelerated teaching program. However the following talks were well received:

The slide rule, by Mr. M. J. Minneman.

Vector analysis (two lectures), by Mr. E. Herzkorn.

Planetary motions, by Mr. J. Diamond.

Operator theory, by Mr. H. Grad.

The officers of the club for 1942–43 were: President, H. Grad; Vice-President, E. Herzkorn; Secretary, D. Frankl; Treasurer, J. Diamond.

Mathematics Club, Mount Mary College

During the past school year, the Mathematics Club of Mount Mary College has met regularly on the second Monday of each month. At these meetings, reports were given on great mathematicians, including

Descartes, by Pat Paisley.

Archimedes, by Therese Rathgeber.

Sir Isaac Newton, by Marge Farrell.

Euclid, by Anne Snyder.

An especially interesting biography of the adviser was presented by Virginia Manly. Meetings also included games, with War Stamps for prizes. To conclude activities for the year, an outing was planned and enjoyed by the club. Officers were: President, Jeanne Garvey; Secretary, Anne Snyder; Treasurer, Mary Jane Crowley; Adviser, Sister M. Felice.

Pi Mu Epsilon, Michigan State College

Although fewer meetings than usual were held during 1942–43, the fall and winter meetings enjoyed a large attendance. At three of the meetings addresses were heard as follows:

Archimedean solids, by Dr. B. M. Stewart.

Synthetic definitions of hyperquadrics, by Dr. L. Toralballa.

Four dimensional geometry, by Professor G. Y. Rainich of the University of Michigan, the address delivered after the annual dinner.

Two of the other meetings featured six or seven brief problems presented by student members, and one meeting was a business meeting. Officers of the chapter were: President, Alice Benedict; Vice-President, Anne Mandenberg; Secretary, John Harrington; Treasurer, Peter Trezize; Faculty Advisers, Dr. J. F. Heyda and Dr. W. L. Mitchell.

Mathematics Club, University of Buffalo

The 1942–43 season opened with a meeting held on October 31 in the Blue Room of the Norton Union, at which new members were welcomed, plans for the year were made, and mathematical games and Hallowe'en refreshments were enjoyed. Topics for the ensuing meetings were:

The probability in dice-throwing games, complemented by actual charts of the data which he had accumulated, by John Castle.

Special mathematical problems, by William Hoctor and Lois Obenauer.

The theory of the game of Nim, by Marjorie Easterbrook, followed by mathematical games.

Measurement of an arc, by John Castle.

The theory of sundials, with directions on how to make one, by Gordon Guernsey.

The last two talks were presented at the annual open-house meeting for high school teachers and students in April, the program for which was planned and conducted by Ruth Brendel. The March meeting was a bowling party to which the Army Air Corps cadets stationed on campus were invited. Officers for the year were: President, Annabel Miller; Vice-President, Jeanne Jerge; Secretary-Treasurer, Lois Obenauer.

Mathematics Club, University of Dayton

Semi-monthly meetings were held at which members of the Club presented papers:

Finite and infinite classes, by Louis Synck.

Do concentric circles intersect?, by John Stang.

The four color map problem, by William McHugh.

Calculating prodigies, by Joseph Overwein.

Why the slide rule works, by Paul Herking.

The game of Nim, by John Westerheide.

At a joint meeting with the engineering seminar group, sponsored by the Club in October, the topic was:

Desirable educational qualifications of an engineer, by Mr. A. E. Schnaitman of the Warner and Swasey Co. of Cleveland. In November the Club sponsored a *Symposium on applied mathematics*, featuring two speakers:

Applications of the calculus of finite differences, by Professor F. M. Tiller of Vanderbilt University.

Fundamental concepts of the calculus of finite differences, by Professor K. C. Schraut.

Other topics presented during the year at the regular meetings were:

The life and work of Leibnitz, by Professor K. C. Schraut.

The method of least squares, by Mr. G. C. Peckham.

At the formal initiation in February, thirty new members were issued certificates. Dr. Francis J. Molz, Associate Dean of the Division of Science, was elected to honorary membership. William Fitzgibbons delivered the vice-president's charge to the new members, addressing them on the subject:

Aims and purposes of the Mathematics Club.

This address was subsequently published in the *Exponent*, which is the University's literary magazine.

Near the end of the year a banquet attended by fifty-five members was held. At the final meeting of the year the Dean of Science First and Second Awards were conferred upon William Fitzgibbons and Joseph Overwein respectively, for having delivered before the Club the most interesting papers of the year. At commencement the *Mathematics Club Alumni Awards for Excellence in Advanced Mathematics* were conferred upon Jack Homan in the Senior Class and Louis Synck in the Junior Class. The officers of the Club were: President, Joseph Overwein; Vice-President, William Fitzgibbons; Secretary-Treasurer, Theodore Schuler. Professor K. C. Schraut was Faculty Adviser.

Mathematical Society, Hofstra College

Speakers and topics for the year were as follows:

Non-Euclidean geometry, by Professor E. R. Stabler.

Denumerability of sets, by Ellen Fletcher.

Probability, by Harry Durham.

Vibration and flutter in aeronautics, by Dr. R. H. Tripp of the Grumman Aircraft Corporation.

At one meeting a quiz contest was held. In May the Club had a joint picnic with the Hofstra chapters of *Kappa Mu Epsilon* and *Beta Beta Beta* (honorary fraternities in mathematics and biology respectively). Officers for the year 1942–43 were: President, Mario Juncosa; Vice-President, Ellen Fletcher; Secretary, Phyllis Reynolds; Treasurer, Harry Durham; Historian, Edward Ryder. The Club Adviser was Professor E. R. Stabler.

Brooklyn College Mathematics Society

The Brooklyn College Mathematics Society adopted a program which attempted to cover the varied interests of the members, and which included the following talks:

War courses in mathematics at Brooklyn College, by Professor MacNeish.

Numeration, by Professor Boyer.

Finite configurations, by Dr. Singer.

Constructions by inversion, by Gerard Washnitzer, a member of the Society.

Linear transformations of polygons, a lecture by Professor Douglas based on his original work.

The Society sponsored the *Math Mirror*, which has been issued annually for the last eleven years, and which featured articles by members of the Society. The Society also sponsored an Integration Contest for the students of calculus at Brooklyn College. The officers for the fall term 1942 were: President, Frances Brand; Vice-President, Julian Kielson; Secretary, Audrey Dirnfeld; Treasurer, Joseph Chiarulli; Publicity Director, Norman Zabb. For the spring term 1943, they were: President, Irving Reiner; Vice-President, Abraham Mark; Secretary, Audrey Dirnfeld; Treasurer, Margot Henn; Publicity Director, Seymour Chays. The Faculty Advisers were Dr. Richardson for the fall term and Mrs. Kormes during the spring term.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SPEED TEST QUESTION. A PROBLEM IN GEOGRAPHY

E. J. MOULTON, Columbia University

How long does it take you to get a correct answer to the following mathematical problem? You may be surprised to know that most mathematicians require at least ten minutes. You are urged to time yourself for a little amusement. Read the problem carefully, solve it mentally, write your answer down, and note the time required. (We add the remark that you are assured that this is a legitimate mathematical problem, in which the earth is assumed to be a sphere.)

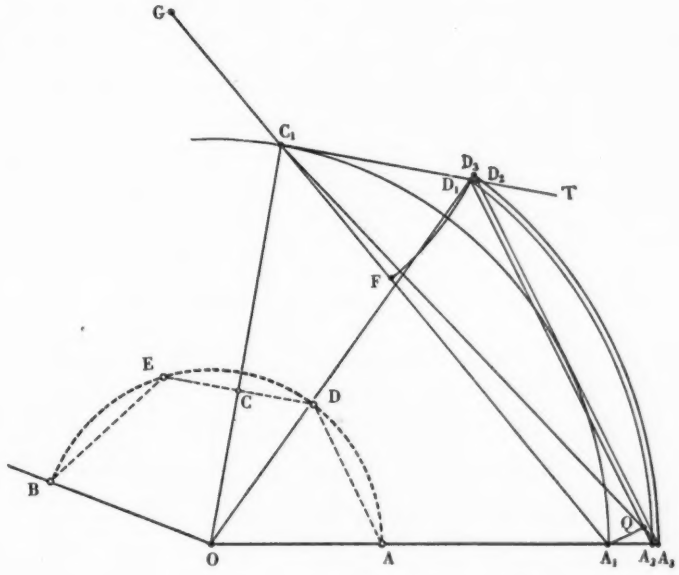
THE PROBLEM. *Starting at a point on the earth's surface, a man walked due south ten miles, then due east ten miles, then due north ten miles, and found that he had returned to the starting point. Where must he have started?*

For comments on the problem see page 220.

A SIMPLE CONSTRUCTION FOR THE APPROXIMATE TRISECTION OF AN ANGLE

R. L. DURHAM, Buena Vista, Virginia

1. Introduction. Since it is impossible to locate the exact trisectors of an arbitrary angle under the terms of Euclid's Postulates alone, the customary methods make use of other curves than circles. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, 10th edition, page 344, or 11th edition, page 333.) There is also the problem of *approximating* to the trisectors with ruler and compasses. Most of the efforts in this direction have been either too crude or too complicated. It seems reasonable to claim, however, that the present construction attains remarkable proximity without undue complication.



For an explanation of the general ideas involved, consider three equal arcs AD, DE, EB of a circle with center O, and let C be the midpoint of the middle arc DE. Then $AD=2DC$. Thus a *given* angle AOB will be trisected by OD if we can locate a point D on the arc AB, twice as far from the point A as from the line OC (which *bisects* the given angle). A hyperbola of eccentricity 2, with focus A and directrix OC, would intersect the circle in such a point D, but that is beyond the prescription "ruler and compasses." Instead, we proceed to construct a first approximation OD_1, and a second approximation OD_2.

In the figure, AOB is drawn as an obtuse angle for the sake of perspicuity. But the method gives a far better approximation when applied to an acute angle.

2. The first approximation. For a given angle AOB, draw a circle with center O (and conveniently large radius), to meet OA and the bisector OC in A_1 and C_1. Draw the tangent C_1T (perpendicular to OC). Then D_1 is defined to be a point on C_1T, twice as far from A_1 as from C_1. It is most easily located by means of a circle of Apollonius, which is the *locus* of points twice as far from A_1 as from C_1. For this purpose, trisect the chord A_1C_1 at F, so that $A_1F=2FC_1$, and take a point G in A_1C_1 (produced) so that $FC_1=C_1G$. Then D_1 is the point where C_1T meets an arc through F with center G.

The line OD_1 fails to trisect the angle. For, although D_1 is twice as far from A_1 as from OC, yet it lies outside the circle through A_1. We now seek a second approximation OD_2, which fails similarly but with a much smaller discrepancy.

3. The second approximation. With center O, draw an arc through D_1 to meet OA_1 (produced) in A_2. With center D_1, draw an arc through A_1 to meet D_1A_2 in Q. Let A_3 be the point of intersection of C_1Q and OA_2 (both produced). With center O, draw an arc through A_3 to meet OD_1 (produced) in D_3. Then the desired point D_2 is where the chord A_3D_3 meets C_1T.

4. Discussion. To see, in a general way, why OD_2 is close to the trisector, we observe that $D_1Q=D_1A_1=2D_1C_1$, whence, by similar triangles, $D_2A_3=2D_2C_1$. Thus D_2 is twice as far from A_3 as from OC. This point D_2, on the chord A_3D_3, lies inside the circle through A_3; but the distance D_2D_3 is small, and the difference OD_3-OD_2 is smaller still, with the result that OD_2 is almost indistinguishable from the true trisector on any ordinary drawing, especially when the angle AOB is small.

To estimate the amount of the discrepancy, let the given angle be 2θ, so that $\theta=\angle A_1OC_1$. We write

$$\alpha = \angle D_1OC_1, \qquad \gamma = \angle D_2OC_1,$$

so that α and γ are the first and second approximations to $\theta/3$. Then we find that

$$3\tan\alpha = (1-\cos\theta)^{3/2}(7+\cos\theta)^{1/2} - \sin\theta,$$

$$\cot\gamma = 2\cos\frac{\theta-\alpha}{2}\csc\theta + \cot\theta.$$

For instance, when $\theta=90°$, γ is about $30°\ 10'$. Again, when $\theta=45°$, γ is about $15°\ 0'\ 12''$. From the nature of this method it appears that the discrepancy $\gamma-\theta/3$ is positive, and diminishes with θ. Thus the construction trisects any acute angle AOB with a maximum error of twelve seconds.

For a more precise investigation we could use the expansions in powers of θ, which begin with

$$\alpha = \frac{\theta}{3} - \frac{7}{6}\left(\frac{\theta}{6}\right)^3 - 25\left(\frac{\theta}{12}\right)^5 + \cdots,$$

$$\gamma = \frac{\theta}{3} + \frac{7}{144}\left(\frac{\theta}{3}\right)^5 - \cdots.$$

A TABLE FOR COMPUTING PERIMETERS OF ELLIPSES

ALAN WAYNE, Rhodes School

The length of a quadrant Q_a of an ellipse of semi-major axis a, semi-minor axis b, and eccentricity k is

$$(1) \qquad Q_a = aE = a\int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{1/2} d\theta, \qquad (0 < k < 1),$$

where E denotes a complete elliptic integral of the second kind. Also,

$$(2) \qquad k^2 = 1 - (k')^2, \quad \text{where} \quad k' = b/a, \qquad (a > b).$$

For given semi-axes, Q_a is usually computed by finding k^2 and using the corresponding value of E from a table [1, pp. 204–205]. A second method is to find arc sin k [1, pp. 2–19] and then E [2, p. 263]. Again, E may be found from tabulated values corresponding to those of $(k')^2$ [3, p. 80].

For more rapid computation, the writer devised the table of values of E corresponding to those of k', as given here.

SEMI-MINOR AXIS	LENGTH OF QUADRANT OF UNIT ELLIPSE $Q_1 = E = Q_a/a$									
$k' = b/a$	0	1	2	3	4	5	6	7	8	9
.0	1.000	000	001	002	003	005	007	009	011	013
.1	016	019	022	025	028	032	035	039	043	046
.2	050	055	059	063	068	072	077	082	087	091
.3	096	102	107	112	117	123	128	134	139	145
.4	151	156	162	168	174	180	186	192	199	205
.5	211	217	224	230	237	243	250	256	263	270
.6	276	283	290	297	304	311	317	324	331	339
.7	346	353	360	367	374	381	389	396	403	411
.8	418	426	433	440	448	455	463	470	478	486
.9	493	501	509	516	524	532	540	547	555	563
1.0	571									

From this table, the perimeter of any ellipse may be found readily, since to each given ellipse with semi-axes a and b there corresponds a "unit ellipse" which has semi-axes 1 and k', and length of quadrant $Q_1 = E$, so that $Q_a = aQ_1$.

For example, to find the perimeter of an ellipse with $a = 3.74''$ and $b = 2.36''$, find $k' = 2.36/3.74 = .631$, and from the table, $Q_1 = 1.298$, so that $P = (4)(3.74)(1.298) = 19.4''$.

With a slide rule the process takes less than a minute. No more than three slide settings are required to find the perimeter to three significant figures, which is adequate for most practical work.

References

1. H. B. Dwight, Mathematical Tables, McGraw-Hill, New York, 1941.
2. R. S. Burington, Handbook of Mathematical Tables and Formulas, Handbook Publishers, Sandusky, Ohio, 1940.
3. E. Jahnke and F. Emde, Tables of Functions with Formulae and Curves, Dover Publications, New York, 1943.

COMMENTS ON THE PROBLEM IN GEOGRAPHY (p. 216)

E. J. Moulton, Columbia University

When this problem is proposed orally to mathematicians, the usual response after a short time, varying from one second to a few minutes, is that *the man must have started at the North Pole.* When a mathematician is then told that his answer is wrong—and this answer *is* wrong—he is likely either to look for some non-mathematical explanation of his error or to become slightly belligerent in defense of his answer. However, he is assured that the problem is a bona fide question in the geometry of a sphere, and that his answer is mathematically incorrect. In spite of years of training in logical thinking, he has slipped on the word "must" in the question, "Where must he have started?" He has found one point where the traveller *may* have started, but there is another point where he may have started.

On second thought, which may last ten minutes or longer, mathematicians discover infinitely many points from which the man may have started; the usual answer then given is, *the man must have started either at the North Pole or at a point on a parallel of latitude which is ten miles north of a parallel of latitude whose circumference is ten miles.* But *this answer* also *is wrong*—the man *may* have started at any one of these points, but you should not assert that he *must* have started at one of them.

A correct (but not very satisfying) answer is that

(1) *he must have started at some point on the earth's surface.*

A more satisfactory answer, which is also correct, is that

(2) *he must have started at some point of a locus S which consists of the North Pole and the circles which are ten miles north of the parallels of latitude whose circumferences are $10/n$ miles, where n ranges over the positive integers.*

This locus S is interesting. It consists of an isolated point and an infinite set of circles which have as a limit circle the parallel of latitude which is ten miles from the South Pole; this limit circle is not a part of the locus.

It is also interesting to see what modifications are required in the preceding discussion if the number ten is replaced by a number x, and to consider the locus S as a function of x on the range $x=0$ to $x=12{,}500$ (calling the circumference of the earth 25,000 miles). I consider this function of x to be one of the most interesting which I have encountered, possessing as it does some startling discontinuities.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Differential and Integral Calculus. Fourth Edition. By C. E. Love. New York, The Macmillan Company, 1943. 15+483 pages. $3.25.

The fourth edition of Love's Calculus has appeared in a new format employing a larger page size and larger type and is illustrated with a set of greatly improved figures.

The textual material has in general been simplified by the inclusion of fuller explanations. The order of presentation of the topics of the Calculus has been changed to include within the first six chapters most of the applications of the differentiation of algebraic functions and the applications of the integration of simple algebraic functions to areas and plane motion. Vectors and parametric equations are also treated within these chapters.

Continuity and discontinuity are discussed in the seventh chapter. There is an advantage in the position of this topic within the text for the teacher who feels that an introduction to the Calculus increases the student's interest in and understanding of such a discussion.

No great changes have been made in the presentation of later chapters. The Integral Test for series has been omitted, and a development of the equation of the catenary has been included. Talks of natural logarithms, hyperbolic functions, and trigonometric functions have been inserted.

FRED KIOKEMEISTER

Geometry with Military and Naval Applications. By W. F. Kern and J. R. Bland. New York, John Wiley and Sons; London, Chapman and Hall, 1943. 7+152 pages. $1.75.

In 1934 the authors of this text in solid geometry produced "solid Mensuration," which was reviewed in this journal (Vol. XLIII, page 232) by R. A. Johnson. A second edition with proofs appeared in 1938. This newest geometry is essentially the 1938 edition dressed up for current use by replacing many of the standard problems with others having a military or naval setting. A few sections of the text have been omitted and 17 pages on trigonometric functions and the solution of right triangles have been added to the appendix.

All three texts are excellent for a course in solid geometry in which the emphasis is on mensuration rather than geometric theorems and their proof. This particular volume will make mensuration more interesting to those students who are in the armed forces or are anticipating early induction.

L. A. DYE

Basic Physics for Pilots and Flight Crews. By E. J. Knapp. New York, Prentice-Hall, Inc., 1943. 118 pages. $1.65.

The author states in his Preface, "The purpose of this book is to teach the fundamentals of physics as a preparation for Meteorology, Theory of Flight, and Engine Operation. It is designed to meet the requirements of primary flight training or air cadet training."

This pocket size handbook presents, in condensed form, certain fundamental principles of Mechanics of Solids and Fluids and of Heat. The following laws are stated and illustrated by problems: Newton's Laws of Motion; Laws of Sliding Friction; Law of Conservation of Energy; Boyle's Law; Charles' Law; Dalton's Law; General Laws of Gases. As a rule these laws are merely stated without reference to experimental data which lead to their formulation.

Problems on addition of vectors are solved by the graphical method only. Subtraction of vectors is not considered. Problems on balanced forces acting at a point in a plane are also solved graphically, without analytical treatment.

Gravitational units are used for the measurement of force. The fundamental equation of motion becomes, consistently, $F=(W/g)\cdot a$.

Discussion of curvilinear motion is limited to the statement that centripetal force equals $(W/g)(V^2/R)$.

Discussion of kinetic energy is limited to kinetic energy of translation. Kinetic energy of rotation is not mentioned.

The subject of elasticity and Young's Modulus is not mentioned.

The student is not warned that the general gas laws hold strictly for an ideal gas only, and that large corrections must be made for easily condensible gases.

These omissions have been made purposely, without doubt, for the sake of condensation. In general, it may be said that definitions and laws are stated accurately; that the text is simply and clearly worded, and that the problems and exercises should be within the capacity of a basic student to solve. On completion of the course, the student should have become familiar with a number of physical formulas and should have learned how to apply these formulas to the solution of simple problems. The handbook may well be useful as a short cut to these objectives. It is not a text on elementary physics.

F. M. GREENLAW

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. With Mathematical Tables. Revised edition. New York, Harper and Brothers, 1943. 14+247+135 pages. $2.40.

In the reviewer's estimation both the present edition and the original version of Nelson and Folley's Trigonometry are excellent examples of what might be called the classical treatment of the subject. Trigonometry done in the traditional manner is, of course, a somewhat crystallized topic; and naturally the various expositions differ, for the most part, but slightly in broad outline. The war, however, has provided an opportunity to freshen this rather stale subject;

and instructors will welcome the fact that the present authors among others have taken advantage of the situation in an effective manner. If one feels the compunction to "follow the book," the teaching of stereotyped freshman courses is apt to be a dull business; and for this reason at least novelty in textbooks should certainly be more welcome than say mere sterile perfection of detail.

The two editions differ in the aspects emphasized, the applications, the tables, and necessarily but unfortunately in format. The first edition was designed primarily to prepare the student for more work in mathematics and the stress is on the functional aspects as opposed to the computational. Thus the analytic portion is completed as soon as feasible; the general definition of the trigonometric functions, to mention one item, is presented at the outset. On the other hand, the second edition, as an impact of the war, places somewhat more emphasis on the computational part than did the earlier exposition. The trigonometric functions are presented first in connection with right triangles, and the solution of oblique triangles is accorded extra weight by early introduction and by the allotment of two chapters to the subject. The first sets of problems on oblique triangles (chapter III) are limited to numbers of two or three digits and hence are well suited to both elementary arithmetic and the slide rule. A later chapter (V) presents the logarithmic treatment. Chapter III is brightened considerably by the introduction of some interesting and timely material on aeronautics which undoubtedly will serve both to relieve the monotony for the instructor and to arouse the student.

In addition to the more common topics there are articles on: significant digits, abridged multiplication, projection, the mil, vectors (addition and resolution into components), aeronautical applications, logarithms (a thorough treatment), graphing by addition of ordinates, complex numbers (elementary operations, polar form, De Moivre's theorem, powers and roots, *etc.*), spherical triangles (thorough and inclusive), applications of spherical triangles (great circle sailing, plane sailing, traverse sailing, middle latitude sailing, dead reckoning, Mercator projection, Mercator sailing, the celestial sphere, the astronomical triangle, determination of latitude and longitude, determination of the time of day and of the time of sunrise and sunset).

The book contains about 155 pages of tables including haversines, logarithms of numbers, logarithms of trigonometric functions, trigonometric functions, natural logarithms, conversions of degrees to radians, squares, cubes, square roots, cube roots, reciprocals, circumferences and areas of circles, and some important constants.

Conclusion. This book is certainly one of those that should be examined by anyone who is considering the selection of a text on the subject. By proper choice of topics and their order it can be made to serve either the war needs for which it was designed or to contribute to the foundation for more mathematics courses. It contains an abundance of problems (2,624), most of which are not in the old edition; and the exposition leaves little to be desired.

H. V. CRAIG

Analytic Geometry. By F. H. Steen and D. H. Ballou. New York, Ginn and Co. 1943. 7+206+9 pages. $2.40.

This book is designed for beginners. A feature of the text is the inclusion of "orals" preceding many of the sets of exercises, to serve as orientation of the purpose of the exercises following. The normal form of the equation of the line and the formula for the distance between a point and a line are derived without use of trigonometry; this method is then followed by a short outline of the traditional one. The area of a triangle in terms of the coordinates of its vertices is given as a determinant, with a foot-note explaining its use. Otherwise, determinants of the third order are not used. The idea of a sign associated with a non-linear equation is not given. The explanation of what the book calls squaring an equation is not helpful. The foot-note concerning imaginaries needs more explanation to be intelligible. The treatment of the circle and of conic sections is clear and concise. It includes that of pencils of lines and of circles. The conics are presented by means of focal distances, but followed by an outline of the method of eccentricity. The use of the complementary equation, page 66, is hazy.

Rotation and elimination of the product term in the equation of a conic are discussed without use of trigonometry, again followed by an alternate method, based on the traditional procedure. A chapter on curve tracing includes symmetry as to an axis, tangents at the origin, asymptotes parallel to the axes and vacant strips parallel to the axes. Trigonometric curves, logarithmic and exponential curves, sums of ordinates and compound curves are presented, the last too sketchily. Parametric representation is discussed and applied to the cycloids; polar coordinates are discussed rather fully, including the tangent at the pole, if the curve goes through the pole.

Coordinates in space are treated in four chapters in the same manner as in the plane. Determinants of the second order are employed in the discussion of angles, but nowhere else. Quadric surfaces are described in a brief outline which includes curves of intersection and projecting cylinders. The book is provided with an index and the answers to the exercises. It has an attractive appearance with an open page, clear type, and over 200 well drawn figures.

VIRGIL SNYDER

Selected Topics in Higher Mathematics for Teachers. Published by the Committee on Higher Mathematics of the Association of Teachers of Mathematics of New York City, 1943. 107 pages. $0.50. (For copies, address the chairman, Max Peters, at Utrecht High School, Brooklyn.)

This book, printed by the photo-offset process, consists of five monographs covering some topics treated in in-service courses sponsored by the above association for the following purposes: first, "to make available to teachers valuable enrichment material drawn from appropriate fields of higher mathematics," second, "to make more profound the teacher's understanding of the concepts he teaches."

The first monograph, on Topology, by Samuel Greitzer, is marred by several obscure statements. For instance, in the derivation of Euler's formula appears the following sentence whose obscurity seems partly due to a misprint and partly to a desire to avoid buckling down to the consideration of all possibilities: "Let us therefore take any polyhedron and reduce it by removing edges and with a simplified figure having one vertex, one face and one or more edges." There is also some tendency to assume unduly knowledge which could not be obtained by reading the monograph. There is, however, a wealth of fascinating illustrations and the presentation is such as to stimulate the reader to look more carefully into the subject.

The second monograph, on Vectors in Plane Geometry, by Harry Sitomer, is a *very* convincing and interesting attempt to show how the use of vectors can simplify certain proofs of plane geometry. The text is interspersed with exercises which are well chosen to teach the reader the concepts involved and the possibilities of the method. The only essential improvement the reviewer could suggest would be to make the figures more accurate—notably the one at the top of page 30.

The third monograph, on the Theory of Numbers in Secondary Mathematics, by Mannis Charosh, presents a good case for the training of teachers in some of the elements of the theory of numbers, though in two instances the author stretches his argument too far: the number tricks on page 42 are explained by algebra and the application to the roots of $\sin x=\frac{1}{2}$ seems somewhat farfetched. The principle types of application given are: finding integer values of F that will make $C=(5/9)(F-32)$ an integer, determining quadratic equations with rational roots, finding integer values of a and b so that the two expressions $x^2+ax\pm b$ have factors with rational coefficients, and determining oblique triangles whose sides and area are integers. The reviewer trusts that even this monograph will not induce teachers to give only problems with integral answers.

The fourth monograph, on the Postulational Approach to Mathematics, by Mannis Charosh, is a good treatment of a difficult subject. A good discussion is given of axiom (self-evident truth) versus postulate. The author's illustration of a mathematical system complete with consistency, independence, and categoricalness serves the purpose admirably.

The fifth monograph, An Introduction to Non-Euclidean Geometry, by Max Peters, describes some notable attempts to avoid the fifth axiom of Euclid and follows this by an introduction to Lobatchewsky's geometry and Riemannian geometry. The monograph is very well written and should be within the understanding of those for whom it is intended.

Though there are a number of misprints, pages 35 and 36 are interchanged, and though in some cases there is a little tendency to be careless in statement and superficial in treatment, it is the reviewer's opinion that this book is an admirable beginning. The fact that the teachers themselves participated in and pushed the venture is very significant and it is to be hoped that this is only the first of a series of such books.

B. W. JONES

Basic Mathematics for Pilots and Flight Crews. By C. V. Newsom and H. D. Larsen. New York, Prentice-Hall, 1943. 6+153 pages. $2.00.

This text provides for the prospective pilot, navigator or bombardier a review of mathematics at the high school level, and offers a wealth of problems designed to stimulate his interest in and understanding of the applications of elementary mathematics associated with the flight of aircraft. The authors state that little of the mathematics offered will be new to the student who has had a year of algebra and a year of geometry.

The reviews of arithmetic, algebra and geometry contained in this text emphasize the application of the subject to the problems of flight. In this respect the discussion of the basic mathematics involved and the illustrations offered differ widely from the usual high school text.

A chapter on elementary vectors and vector diagrams with applications and problems is provided. A chapter is devoted to the circular slide rule and its use in navigation. Answers are provided for about half of the problems.

While this text is essentially at the high school level the point of view taken by the authors in their discussions together with the nature of the applications makes this text a valuable one for the prospective student of aeronautics.

H. N. HUBBS

Solid Geometry and Spherical Trigonometry. By H. C. L. Leighton. New York, D. Van Nostrand Co., 1943. 19+192 pages. $2.20.

The contents of this book, as indicated by the title, are preceded by a summary of the contents of plane geometry, including the axioms, postulates and an outline of those theorems which are not mentioned again in the body of the text. Among the added postulates of the new part are featured that:

a plane is determined by a line and a point not on the line;

if two planes intersect, their intersection is a straight line.

The scheme of instruction is to present a list of pertinent questions for orientation, without providing proofs for all of them. This requires a large number of figures; each part of a complicated demonstration is featured in this way. Special attention is given to Napier's Rules for the right spherical triangle. It is pointed out that any spherical triangle can be broken up into right spherical triangles, each of which could then be discussed in this way. A later chapter derives the law of sines and the law of cosines in the usual manner. A short chapter on applications includes a description of true bearing, great circle sailing, and the Mercator chart. A final chapter provides the derivation of the half-angle formulas; they are not used in the earlier part. The printing and press-work are excellent. A clear, open page is presented, and the figures, nearly three hundred of them, are well done.

VIRGIL SNYDER

Business Mathematics. By C. L. Richtmeyer and J. W. Foust. New York, McGraw-Hill Book Company, second edition, 1943. 15+401 pages. $2.75.

The first edition of this useful text for students of business mathematics appeared in 1936 (see review this MONTHLY, Vol. XLIII, No. 9, p. 570). The book has now been considerably enlarged, the present edition containing 318 pages, exclusive of tables, answers, etc., as compared with 182 pages originally.

The organization and treatment in the first edition are retained in the second, but certain changes have been made which increase the usefulness of the book. Among these changes are:

1. The material on elementary arithmetic has been expanded. The work on integers and fractions which was treated in one chapter in the first edition occupies three in the second.

2. Self-tests which appeared only in the first chapter of the earlier edition now occur at the end of each chapter. These tests should prove to be a valuable feature of the book.

3. Lists of supplementary references are given at the end of each chapter.

Other changes include expansion of the chapter on graphs, computations with approximate numbers, and installment buying. The work on bar graphs is more complete and includes several additional illustrative problems. Approximate numbers are introduced early in the text and are used to good advantage throughout the book. Also, a short account of correlation is included in the chapter on statistics. Problem and exercise lists have been revised and new problems added.

R. G. PUTNAM

Mathematics of Flight, By James Naidich. New York, McGraw-Hill Book Co., 1943. 6+409 pages. $2.75.

The purpose of this text is to give the necessary foundations to a student who wishes to fly, build, or analyze the mechanics of airplanes. The goal is all-embracing and the results are correspondingly lacking in depth.

The mathematical content is of high-school level and includes the barest essentials of arithmetic, algebra, plane geometry, and trigonometry. There is a tremendous effort to drill the student on these essentials and their applications to simple problems in the theory of airplanes. Such an effort could be justified only with a very young student, with one of low intelligence, or with one of very weak background.

The aerodynamical content consists in a few of the basic concepts such as center of gravity, stability, lift, drag, angle of attack, and banking, with purely intuitive explanations and great emphasis on practical applications. Much space is given to pictures of new models of aircraft. The presentation of these topics is excellent and should be quite successful in giving a young student a feeling for the subject.

W. KAPLAN

Original Tables to 137 Decimal Places of Natural Logarithms for Factors of the Form $1 \pm n \cdot 10^{-p}$, Enhanced by Auxiliary Tables of Logarithms of Small Integers. By H. S. Uhler. Published by the author, New Haven, Conn., 1942. 21+97 pages.

This is a set of radix tables similar to, but larger in extent than, those of Wace, Flower, and others. Their preparation was stimulated by the author's desire to extend his tables of $1/n!$ (Trans. of Conn. Acad. Arts and Sci., vol. 32, 1937) to larger n. Table I gives the logarithms of the integers 2, 3, $\cdots$, 10 and 10^{10^k}, $k=2, 3, 4, 5, 6, 7, 8, 9, 11$ to 137 places. Table II gives the logarithms of numbers of the form $1-n \cdot 10^{-p}$ for $n=1, 2, \cdots, 9$; $p=1, 2, \cdots, 69$ to 137 places, while Table III gives the logarithms of numbers of the form $1+n \cdot 10^{-p}$ for $n=1, 2, \cdots, 9$; $p=1, 2, \cdots, 21$. Tables V and VI are skeleton forms for Tables II and III for $p>20$. Finally, Table IV gives the values of $\log_e 10$ and $\log_{10} e$ to 325 places as well as logarithms of the primes from 11 through 113 to 148 places.

The method outlined by the author in his introduction to find the logarithm of a number N is essentially the one usually attributed to Weddle. N is multiplied by a preliminary reduction factor N' and then by a sequence of factors of the form $1 \pm n \cdot 10^{-p}$ to produce a number $1 \pm x$ where x is so small that $\log (1 \pm x)$ can easily be evaluated to the desired accuracy by the series expansion for $\log (1 \pm x)$. Since logarithms of the factors $1 \pm n \cdot 10^{-p}$ are known and $\log N'$ can be found with the aid of Tables I and IV, $\log N$ can be calculated. To find an antilogarithm, the process is reversed by first finding the factored number and then expanding the factors, *i.e.*, using the method known by the name of Hearn.

The introduction contains several examples of both procedures, the method used to calculate the tables, and checks used to insure complete accuracy.

D. E. RICHMOND

Analytic Geometry. By E. E. Smith, Meyer Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc., 1943. 12+298 pages. $2.50.

This text is not for a brief course in analytic geometry, although the authors point out that certain sections may be omitted in order to enable it to serve in that capacity. It contains sufficient material for a full year's course of three hours a week or a semester's course of five or six hours a week. There are fifteen chapters (202 pages) of plane analytic geometry and five chapters (66 pages) of solid analytic geometry. These are followed by an appendix containing useful formulas from geometry, algebra, and trigonometry; the Greek alphabet; the proof of the invariance of Δ in connection with the general conic; and tables of trigonometric functions, logarithms, and values of e^x and e^{-x}. Answers to odd-numbered problems are listed.

One is impressed by the excellent figures in the book. They are so clearly marked that all the required features are unmistakably indicated.

Throughout the book discussions of general equations precede those of special cases. This is particularly noticeable in the treatment of the general

equation of the second degree in both plane and solid analytic geometry. It makes for a considerable amount of unity since each particular special case is referred back to the general equation for comparison.

Although the topics considered are those generally treated in any book on the same subject, there are many novel features in the presentation. Among these are the less usual constructions given for conics; the use of the terms normal axis, normal angle and normal intercept in connection with the straight line; the discussion of factorable equations; the definition of an asymptote; the derivation of the equation of a conic through five given points; the use of slope as well as other devices in curve tracing; the determination of parametric equations from a given Cartesian equation; and the use of the term direction parameters rather than direction numbers or direction components.

Topics sometimes slighted in other texts are treated at greater length here. For example, curve tracing in polar coordinates is discussed from the standpoint of intercepts, extent and symmetry in a manner analogous to that used for Cartesian coordinates. Some books give the impression that the treatment for polar coordinates is entirely different from the other. Empirical equations are given a generous amount of attention, using the method of average equations rather than the method of least squares. The treatment of tangents, normals and diameters is quite complete. The thoroughness of these discussions is accomplished, however, without emphasis on unimportant details. In fact, many steps in the derivations of formulas are omitted and there is frequent use of the expression "it may be shown that." There is wisdom in these omissions in that the overall pattern of the discussion is more evident, but they might prove difficult for the poorer student.

Determinants are used throughout the book with only a brief reference to them in the list of useful formulas. Students should really have a knowledge of determinants in order to use the book to the best advantage.

There are a few typographical errors, particularly in the small type material, but no glaring errors were detected. Difficulties of war time printing may be blamed for the majority of defects. On the whole the authors have developed a sound text which should take its place among the better books on the subject.

HARRIET F. MONTAGUE

Air Navigation for Beginners. By S. G. Lamb. New York, Norman W. Henley Publishing Co., 1942. 103 pages. $1.50.

This small volume was designed for the young student who knows nothing about navigation and little about mathematics. Neither trigonometric functions nor logarithms are used. The problems considered require only arithmetic or simple geometrical constructions. Teachers who prefer to have their students use a method first and find out why it works afterward will be disappointed that the author mentions the Dalton Dead Reckoning Computer, the Nautical Almanac, and Hydrographic Office Publications No. 211, 208, and 214 without giving the reader any idea of the way in which these navigational aids are used.

If one is willing to accept the author's approach, one will find little to criticize in the material included. The first two chapters cover fundamentals such as latitude, longitude, direction and distance on the earth. Chapter III describes the simpler instruments of navigation. At no point in the book will the reader find the sextant, the gyro-compass, and other complex instruments described. Chapters IV to VII cover briefly but fairly adequately Maps and Charts, Piloting and Radio Bearings, Dead Reckoning, The Wind Triangle and Wind Triangle Applications. Chapter VIII describes the radio range system of the Civil Aeronautics Administration as it existed in the United States before the war.

Chapters IX, X, and XI concern Flight Instruments and Corrections, Aerial Astronomy, and Time. There follows twelve pages of questions and answers and one page of abbreviations.

This book may be recommended as an introduction to air navigation for high school students.

C. H. SMILEY

NEW BOOKS RECEIVED

An Introduction to Pure Solid Geometry. By G. S. Mahajani. Second Edition. Poona, (4), Aryabushan Press, 1943. 10+104 pages. Rs. 3-0-0.

Lessons in Elementary Analysis. By G. S. Mahajani. Third Edition. Poona (4), Aryabushan Press, 1942. 13+298 pages. Rs. 6-4-0.

An Intermediate Course in Differential Equations. By E. D. Rainville. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd. 6+213 pages. $2.75.

Geometry with Military and Naval Applications. By W. F. Kern and J. R. Bland. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Inc., 1943. 7+152 pages. $1.75.

Statistical Adjustment of Data. By E. W. Deming. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 10+261 pages. $3.50.

Elements of Trigonometry. Plane and Spherical. By L. M. Kells, W. F. Kern, J. R. Bland, and J. B. Orleans. New York and London, McGraw-Hill Book Co., Inc., 1943. 10+363 pages. $1.80.

Plane and Spherical Trigonometry. By D. H. Ballou, and F. H. Steen. Boston, Ginn and Company, 1943. 6+179+9 pages. $2.20.

Spherical Trigonometry with Tables. By D. H. Ballou and F. H. Steen. Boston, Ginn and Company, 1943. 4+68+84 pages. $1.25. (This is latter half of Plane and Spherical Trigonometry, pages 109–176 published with Tables.)

Principles of Air Navigation. By B. A. Shields. New York and London, McGraw-Hill Book Co., Inc., 1943. $2.20.

Grösse Masszahl und Einheit. By M. Landoldt. Zürich, Rascher and Co., 1943. Fr. 5.50.

Mathematics for Mariners. By C. E. Dimick and C. C. Hurd. New York, D. Van Nostrand Company, Inc., 1943. 7+254 pages. $2.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 616. *Proposed by J. S. Cromelin, Chicago*

A 60′ ladder and a 77′ ladder rest with their lower ends against a building on the east side of a street, their upper ends against the opposite building on the west side. A third ladder rests with its lower end against the building on the west side, its upper end against the building on the east side. It crosses the first ladder at a height of 17′, the second at a height of 19′. What is the length of the third ladder, and how wide is the street (to the nearest inch)?

E 617. *Proposed by V. Thébault, San Sebastián, Spain*

From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if and only if the second tetrahedron is trirectangular.

E 618. *Proposed by Thorold Gosset, Cambridge, England*

Prove that, for any value of θ between 0 and π,

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \frac{\pi}{4}.$$

E 619. *Proposed by W. B. Clarke, San José*

Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles.

E 620. *Proposed by Alan Wayne, Rhodes School, New York*

Find integral sides for a triangle in which one angle is six times another.

SOLUTIONS

Two Remarkable Squares

E 582 [1943, 454]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a square of ten digits such that the two numbers formed by the first five and last five digits are consecutive.

Solution by D. H. Browne, Buffalo, N. Y. Instead of the equation

$$x^2 = abcde \cdot 100001 \pm 1,$$

consider the congruence

$$x^2 \equiv \pm 1 \pmod{11 \cdot 9091}.$$

Since -1 is a quadratic non-residue (mod 11), the upper sign must be taken. Thus $x-1$ and $x+1$ must be multiples of 11 and 9091 (or vice versa) which differ by 2, say

$$11u - 9091v = \pm 2.$$

Since $9091 \equiv 5 \pmod{11}$, we have $5v \equiv \mp 2$, whence $v \equiv \pm 4 \pmod{11}$. Among such values of v, 4 and 7 are the only ones for which $9091v$ has five digits. Hence the two solutions

$$36365^2 = 1322413225, \qquad 63636^2 = 4049540496.$$

Also solved by R. L. Caswell, Daniel Finkel, N. G. Gunderson, Irving Kaplansky, Walter Penney, E. D. Schell, E. P. Starke, and the proposer.

Radical Axes of Spheres

E 583 [1943, 454]. *Proposed by N. A. Court, University of Oklahoma*

Given four spheres (A), (B), (C), (D), with centers A, B, C, D, let a plane parallel to ABC cut DA, DB, DC in points U, V, W. Show that the radical axis of the three spheres having U, V, W for centers and coaxal with the respective pairs of spheres (D) and (A), (D) and (B), (D) and (C), coincides with the radical axis of the spheres (A), (B), (C).

Solution by P. D. Thomas, Wright Junior College, Chicago. A sphere orthogonal to two spheres is orthogonal to any sphere of the pencil determined by them. Hence the radical sphere of (A), (B), (C), (D), being orthogonal to these, is also orthogonal to (U), (V), (W), and coincides with the radical sphere of (U), (V), (W), (D). Its center lies on the radical axis of (A), (B), (C), and on the radical axis of (U), (V), (W). Hence these two radical axes, perpendicular to the parallel planes ABC and UVW, must coincide.

Also solved by Howard Eves and the proposer. The corresponding proposition in the plane is due to C. Servais (*Mathesis*, 1891, p. 238, q. 717).

Curve of Flight by Directional Radio

E 586 [1943, 512]. *Proposed by Frank Hawthorne, Allegheny College*

What is the path of an airplane "homing" from A to B by directional radio from B only, if acted on by a constant wind?

Solution by W. N. Brown, Glendora, California. Use polar coordinates with origin at B and initial line in the direction of the wind. Let (a, α) be the given position A, and (r, θ) a subsequent position. If v is the constant velocity of the airplane relative to the air, and w is the constant velocity of the wind, the resultant velocity, along the tangent to the path, is

$$\frac{dr}{rd\theta} = \frac{v + w\cos\theta}{w\sin\theta}.$$

Integrating, and evaluating the constant, we obtain the polar equation

$$r = a\frac{\tan^{v/w}\frac{1}{2}\theta\sin\theta}{\tan^{v/w}\frac{1}{2}\alpha\sin\alpha}.$$

Also solved by Howard Eves, referring to Agnew's *Differential Equations* (New York, 1942), p. 88.

Altitudes of an Orthocentric Tetrahedron

E 587 [1943, 512]. *Proposed by V. Thébault, San Sebastián, Spain*

Let AA', BB', CC', DD' be the altitudes of an orthocentric tetrahedron $ABCD$, with orthocenter H. Show that

$$\frac{BC\cdot DA}{B'C'\cdot D'A'} = \frac{CA\cdot DB}{C'A'\cdot D'B'} = \frac{AB\cdot DC}{A'B'\cdot D'C'} = \frac{HA\cdot HB}{HC'\cdot HD'} = \frac{HC\cdot HD}{HA'\cdot HB'} = \cdots.$$

Solution by Howard Eves, Syracuse University. Since the four points A, A', B, B' are concyclic,

$$AB/A'B' = HB/HA' = HA/HB'.$$

Similarly $DC/D'C' = HC/HD' = HD/HC'$. Therefore

$$\frac{AB\cdot DC}{A'B'\cdot D'C'} = \frac{HB\cdot HC}{HA'\cdot HD'} = \frac{HB\cdot HD}{HA'\cdot HC'} = \frac{HA\cdot HC}{HB'\cdot HD'} = \frac{HA\cdot HD}{HB'\cdot HC'}.$$

Similar results for the other pairs of opposite edges establish the theorem.

Also solved by L. M. Kelly and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4115. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

From a point P on the circumcircle of a triangle lines are drawn inclined at angles θ to the sides of the triangle and meeting them in three collinear points. Prove that as P varies, the line on the three points envelops a three-cusped

hypocycloid. Prove that this hypocycloid is the locus of a point on a circle, radius $R/2 \sin \theta$, which rolls inside another circle, radius three times that of the rolling circle, whose center X is equidistant from the circumcenter C and the orthocenter O, and is such that angle $OXC = 2\theta$, R being the circumradius.

4116. *Proposed by N. A. Court, University of Oklahoma*

Given the tetrahedron $(T_1) = SA_1B_1C_1$, the tangent plane to its circumsphere at the diametric opposite of S meets the edges SA_1, SB_1, SC_1 in the points A_2, B_2, C_2. The tangent plane to the circumsphere of the tetrahedron $(T_2) = SA_2B_2C_2$ at the diametric opposite of S meets the edges of (T_2) through S in the points A_3, B_3, C_3 thus forming the tetrahedron $(T_3) = SA_3B_3C_3$, *etc.* Find the locus of the incenters of these tetrahedrons.

4117. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A polygon $A_1A_2 \cdots A_n$ may be transformed into a polygon $B_1B_2 \cdots B_n$ by locating the points B_i on the sides A_iA_{i+1} so that the ratio of A_iB_i to B_iA_{i+1} is equal to a constant r. Prove that if T_1 and T_2 are two such transformations for the ratios r_1 and r_2, then $T_1 \circ T_2 = T_2 \circ T_1$, and generalize.

4118. *Proposed by Otto Dunkel, Washington University*

Show that

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3)\cdots n}{6!8} [15n^3 + 30n^2 + 15n - 2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

4119. *Proposed by V. Thébault, San Sebastián, Spain*

The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the respective opposite sides meet in a point P. Show that the six points of contact of circles tangent to two sides and orthogonal to a given circle with center P are on a circle concentric with the inscribed circle.

SOLUTIONS

An Inequality for Triangles

4070 [1943, 124]. *Proposed by P. Erdös, Princeton, N. J.*

Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho + r \leqq m$.

Correction. The proposer intended to exclude obtuse angled triangles.

I. *Solution by L. A. Santaló, Rosario, Argentina.* Let $A_1A_2A_3$ be a triangle with no obtuse angle and with angles $A_1 \geqq A_2 \geqq A_3$; let O, I, H_i, B_i be the circumcenter, incenter, foot of the altitude from A_i, point of contact of incircle (I)

with side $a_i = A_jA_k$. If $I \equiv O$ the triangle is equilateral and $\rho + r = m$. If $A_1 = \pi/2$ and $A_1A_2 = A_1A_3$, then $\rho + r = m$.

Since $A_2B_1 \leqq B_1A_3$ and angle $OA_3B_1 \leqq$ angle IA_3B_1, the center O does not lie outside the triangle B_1A_3I. Also A_3O and A_3H_3 are symmetric with respect to A_3I. Let O_3 on A_3H_3 be the symmetric of O with respect to A_3I so that $A_3O_3 = r$, and let I_3 be the orthogonal projection of I on A_3H_3 so that $I_3H_3 = \rho$. Since angle $O_3IB_3 > \pi/2$, the point O_3 lies on segment A_3I_3, and it follows that $\rho + r \leqq A_3H_3 = m$.

If $A_1A_2A_3$ has an obtuse angle it is not easy to determine the relation between m and $\rho + r$. If $A_1A_2 = A_1A_3$ and angle $A_1 \to \pi$, then $m \to 0$; hence there are triangles for which $\rho + r > m$. On the contrary, if $A_1 > \pi/2$, $A_1A_2 < A_1A_3$ with A_1A_3 fixed, and we let $A_1 \to \pi/2$ and $A_1A_2 \to 0$; then we see that there are obtuse angled triangles for which $\rho + r < m$.

II. *Solution by Alfred Brauer and I. S. Cohen, University of North Carolina.* It is easy to see that this theorem is not always true. For example, let the triangle be isosceles and let its vertex angle approach 180°. Then $\rho \to 0$, $r \to \infty$, $m \to 0$, so that $\rho + r \leqq m$ cannot hold. The proposer subsequently indicated that the triangle should have been acute.

In the following, we prove, in fact, considerably more: Let the angles A, B, C of the triangle be such that

$$A \leqq B \leqq C. \tag{1}$$

Then the above theorem is true if $B \geqq 45°$; on the other hand it is definitely false if $B < 2$ arc sin $\frac{1}{2}(\sqrt{3} - 1) = 42°\ 56'+$. If $42°\ 56'+ < B < 45°$, then the theorem may be true or false, and it is definitely false if also $C \geqq 135°$. For an acute triangle, the theorem then follows from the fact that then $B \geqq 45°$.

Since A is the smallest angle, the longest altitude will be the one drawn from the vertex A. It can be shown that

$$\frac{m - \rho - r}{r} = \cos(C - B) - \cos C - \cos B = f(B, C).$$

It follows from (1) that

$$B \leqq C, \qquad B + C < 180°, \qquad 2B + C \geqq 180°.$$

These inequalities define in the (C, B)-plane a certain triangular domain, and we are interested in the values of $f(B, C)$ in this domain.

For a fixed B $(0 < B < 90°)$, let $f(B, C)$ be considered as a function of C in $B \leqq C \leqq 180°$. Then

$$\frac{\partial f}{\partial C} = -\sin(C - B) + \sin C$$

vanishes if and only if

$$C = 90° + \tfrac{1}{2}B,$$

and it is easily verified that this gives a maximum of f; moreover this is the only maximum in $B \leqq C \leqq 180°$ (for fixed B).

We now distinguish the cases $B \geqq 60°$ and $B < 60°$. If $B \geqq 60°$, then for the significant values of C we have

$$B \leqq C \leqq 180° - B \leqq 90° + \tfrac{1}{2}B.$$

Therefore, for fixed B, $f(B, C)$ is increasing in the significant interval, and so $f(B, C) \geqq f(B, B) = 1 - 2\cos B \geqq 0$. Thus the theorem is proved when $B \geqq 60°$.

If, now, $45° \leqq B < 60°$, then the significant values of C are defined by $180° - 2B \leqq C < 180° - B$. Then the maximum $C = 90° + \frac{1}{2}B$ lies in this interior of the interval, and we must consider the function at both endpoints. Now at the left endpoint, $f(B, 180° - 2B) = \cos 2B(1 - 2\cos B) \geqq 0$. At the right endpoint, $f(B, 180° - B) = -\cos 2B \geqq 0$. The theorem is now proved for all triangles for which $B \geqq 45°$.

To see when the theorem will not be true, we note that it will certainly be false for those values of B for which $f(B, C)$ is negative at the maximum $C = 90° + \frac{1}{2}B$. At this maximum we have $f(B, 90° + \frac{1}{2}B) = 2\sin^2 \frac{1}{2}B + 2\sin \frac{1}{2}B - 1$. Since the roots of the quadratic $2x^2 + 2x - 1$ are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$, we have $f(B, 90° + \frac{1}{2}B) < 0$, if $-\frac{1}{2} - \frac{1}{2}\sqrt{3} < \sin \frac{1}{2}B < -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, that is, if

$$B < 2 \arcsin \tfrac{1}{2}(\sqrt{3} - 1) = 42°56' +.$$

Thus the theorem is certainly false if B is less than this angle.

If $42°\ 56' + \leqq B < 45°$, the theorem may be true or false, depending on the value of C. We show that if $C \geqq 135°$, then it is false. Namely,

$$f(B, C) = \cos (C - B) - 2\cos \tfrac{1}{2}(C + B)\cos \tfrac{1}{2}(C - B).$$

Since $C \geqq 135°$, B is $< 45°$, it follows that $C - B > 90°$, and $\cos (C - B) < 0$. Since $\frac{1}{2}(C + B)$ and $\frac{1}{2}(C - B)$ are between 0° and 90°, the second term is also negative.

Solved also by H. Eves and I. Kaplansky.

Editorial Note. The solution by Kaplansky used the function $f(B, C)$ above with somewhat similar results. Eves showed that the theorem is not true for all obtuse triangles; and, for right and acute angled triangles, he gave a synthetic proof based on the equality $x_1 + x_2 + x_3 = \rho + r$ given in Johnson's *Modern Geometry*, art. 298, f. where the x_i's are absolute normal coordinates of the circumcenter of any triangle $A_1A_2A_3$. Using this equality and the relation $h_ia_i = a_1x_1 + a_2x_2 + a_3x_3$, where a_i and h_i are lengths of sides and corresponding altitudes, we easily obtain a proof different from that of Eves. If $a_1 \leqq a_2 \leqq a_3$, we have at once $h_3 \leqq \rho + r \leqq h_1$. Returning to the function $f(B, C)$ there are two angles C_1, C_2 for which $f(44°, C) = 0$ which are approximately 95° 45.8′, 128° 14.2′, and $f(44°, C)$ is positive or negative according as C lies within or outside the interval from C_1 to C_2.

Evaluation of a Determinant

4071 [1943, 124]. *Proposed by Harry Langman, Brooklyn, N. Y.*

Setting

$$\frac{1}{1^{t+1}}+\frac{1}{2^{t+1}}+\frac{1}{3^{t+1}}+\cdots+\frac{1}{n^{t+1}}=b_t,$$

show that

$$\begin{vmatrix} b_0, & -b_1, & b_2, \cdots, (-1)^{n-1}b_{n-1} \\ -(n-1), & b_0, & -b_1, \cdots, (-1)^{n-2}b_{n-2} \\ 0, & -(n-2), & b_0, \cdots, (-1)^{n-3}b_{n-3} \\ \cdot & \cdot & \cdot \quad \cdots \quad \cdot \\ 0 & 0 & 0 \quad b_0 \end{vmatrix} = 1.$$

Solution by W. A. Bowers, Cornell University. Consider the n numbers $\beta_1, \beta_2, \cdots, \beta_n$; let s_k denote the sum of their kth powers; and p_k the elementary symmetric function involving k factors. Then by Newton's formula, see Dickson, *Elementary Theory of Equations*, p. 70, we have

$$s_k - p_1 s_{k-1} + p_2 s_{k-2} - \cdots + (-1)^{k-1} p_{k-1} s_1 + (-1)^k k p_k = 0, \quad k = 1, 2, \cdots, n.$$

Solving these equations for p_n by Cramer's rule and making certain simplifications and rearrangements in the determinants, we have

$$\begin{vmatrix} s_1 & -s_2 & s_3 \cdots (-1)^{n+1}s_n \\ -(n-1) & s_1 & -s_2 \cdots (-1)^n s_{n-1} \\ 0 & -(n-2) & s_1 \cdots (-1)^{n-1}s_{n-2} \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdots \quad -s_2 \\ \cdot & \cdot & \cdot \quad \cdots \quad s_1 \end{vmatrix} = n! p_n.$$

We obtain the desired result by setting $\beta_k = 1/k$ and $s_k = b_{k-1}$. The result is so easily obtained that I should think it could be found in the literature.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A volume on mathematics in honor of Professor J. Rey Pastor is being published by Rosario University, Argentina. Invited contributors from the United States are G. D. Birkhoff, Jaime Lifshitz, M. S. Vallarta, George Polya, Arthur Rosenthal, Edward Kasner, Arthur Korn, Jacques Hadamard, John DeCicco, J. V. Uspensky.

A portrait of Professor Emeritus R. C. Archibald, given by a number of his friends, was presented to Brown University.

Professor V. W. Adkisson of the University of Arkansas has been appointed chairman of the department of mathematics.

Assistant Professor A. T. Brauer of the University of North Carolina has been promoted to an associate professorship.

Dr. W. S. Erickson of St. Olaf College, Northfield, Minnesota, has been appointed to a professorship at Minot State Teachers College, Minot, North Dakota.

Assistant Professor R. N. Haskell of the University of Texas has been promoted to an associate professorship.

Assistant Professor G. E. Hay of the University of Michigan has been granted leave of absence to work on an NDRC project at Brown University.

Dean H. M. Hosford of the College of Arts and Sciences of the University of Arkansas has been appointed vice-president.

Professor R. L. Jeffery of Acadia University has been appointed to a professorship at Queen's University, Kingston, Ontario.

Professor H. W. March and Assistant Professors Churchill Eisenhart and Stanislaw Ulam of the University of Wisconsin are on leave for war research.

Dr. J. C. C. McKinsey of New York University has been appointed to an assistant professorship at Montana State College.

Professor F. D. Murnaghan of The Johns Hopkins University has been appointed to a visiting professorship at Brown University for the current year.

Professor C. V. Newsom of the University of New Mexico has been appointed head of the department of mathematics at Oberlin College beginning July 1, 1944.

Assistant Professor J. F. Randolph of Cornell University has been appointed to a professorship at Oberlin College.

Dr. Henry Scheffé of Princeton University has been appointed to an assistant professorship at Syracuse University.

Colonel R. H. Somers (U. S. Army retired) has been appointed visiting lecturer in mathematics at Dartmouth College.

Dr. G. L. Walker of the University of Delaware has been promoted to an assistant professorship and has been granted leave of absence to teach at Cornell University.

Associate Professor G. A. Williams of Oregon State College has been promoted to a professorship.

Dr. Y. K. Wong of the University of Chicago has been appointed lecturer at the University of North Carolina.

The following appointment to an instructorship is announced:
Dartmouth College: D. B. Kirk.

Professor Emeritus T. S. Fiske of Columbia University died January 10, 1944.

Professor John Matheson of Queen's University, Kingston, Ontario, died January 24, 1944. He was a charter member of the Mathematical Association.

Professor T. R. Rosebrugh of the University of Toronto died January 24, 1943. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

DEFERMENT OF INSTRUCTORS IN THE NAVY COLLEGE PROGRAM

The following memorandum was issued February 8, 1944, by the Office of L. E. Denfeld, Acting Chief of Naval Personnel. Schedule "A" referred to in the bulletin has been sent to the presidents of all universities and colleges participating in the Navy College Program.

"College officials charged with the administration of the Navy College Program should take all steps necessary to prevent the loss of needed members of their instructional staffs through induction into the Armed Forces.

(a) They should request all instructors to file their qualifications with the National Roster of Scientific and Specialized Personnel, Washington, D. C., since the Roster is utilized for informational purposes by the Selective Service system.

(b) They should file Selective Service Form 42A with the Local Board, in all appropriate cases, and request the deferment of all instructors considered essential.

(c) Where replacement of personnel is contemplated but cannot be effected immediately, they should prepare a Replacement Schedule in accordance with Local Board Memorandum No. 158-A. Intra-term replacements should not be scheduled. The Navy Liaison Officer at State Headquarters is authorized to assist in preparation.

(d) Where the request for the deferment is refused by the Local Board, they should appeal within the time allowed—ten days from the date the Local Board

mails the notice of reclassification to the registrant. Transfer of this appeal to the Board where the instructor is employed should be requested.

(e) Where the appeal has been dismissed by the State Director, they should communicate with the Bureau of Naval Personnel, Navy Department, and request the Bureau to make representations at the Washington level. In order that the Bureau may have the necessary facts, the attached schedule 'A' should be completed and forwarded with such requests via the Naval Officer in Charge for his endorsement and transmission to the Bureau of Naval Personnel.

"The following policies govern the disposition of requests by institutions to obtain the services of former instructors now members of the Navy:

(a) Instructors, now members of the Navy, on active duty, will not be placed on an inactive status so that their services may be utilized by educational institutions.

(b) Instructors, now members of the Navy, will not be detailed to an active duty status to perform teaching or administrative assignments under their former employer.

(c) In the case of the irreplaceable teacher whose induction has taken place, causing serious jeopardy to the Navy V-12 Program, institutions may submit a full statement of the facts to the Bureau of Naval Personnel, via the Officer-in-Charge of the Navy Unit at the institution, for consideration of methods of remedial action. The statement of facts should include the following information in addition to that covered by the attached schedule 'A':

Date of Induction_____
Service to which Assigned_____
Service Number_____

The statement of fact should include a certification by the President of the College that the teacher has been employed and would in the future, if available, be employed full time in connection with the training of naval personnel (or members of the Armed Forces, giving detailed explanation) on active duty, and that no other member of the faculty or college staff is available and qualified to replace the teacher.

"The principles outlined herein for Navy V-12 Units apply equally to Naval Flight Preparatory Schools."

DESCRIPTION OF THE PROFESSION OF MATHEMATICS

The following "description of the profession of mathematics" has been distributed by the National Roster of Scientific and Specialized Personnel to offices of the United States Employment Service.

Occupational Summary

The greatest percentage of the personnel in this profession is engaged in teaching mathematics at the college and university level. A relatively large staff is required since the training is basic to most sciences, pure and applied. The mathematical subjects taught include algebra, calculus, trigonometry, geometry,

differential equations, and applied mathematics. There is a present trend toward the wider utilization of mathematicians in industry, although not many mathematicians are now employed as such. Teachers of mathematics at high school level are not always sufficiently trained to be considered mathematicians, and the scope of their teaching is generally limited to elementary algebra and geometry.

Major Branches

Since most mathematicians are engaged in teaching the subjects enumerated above, only the small non-teaching group specialize in the subdivisions listed below.

1. *Analysis.* Industrial mathematicians are usually employed in a consultative capacity to a research engineering staff. As such, they are called upon to advise concerning mathematical analytical methods for solving engineering problems. These methods include the use of calculus, differential equations, and vector and tensor analysis.
2. *Applied Mathematics* includes its applications in the field of ballistics, cryptanalysis, harmonic analysis and in probability and mathematical statistics. The latter specialty involves the gathering of data by sampling methods, evaluating such information, and preparing it for the guidance and use of technicians in related fields.
3. *Other Specialized Areas* include the theory of numbers, special functions, and the advanced areas of geometry—differential, projective and topologic.

Functional Specializations

As indicated in the occupational summary, mathematicians specialize in any one or more of the following areas.

1. Research
2. Teaching (often combined with research)
3. Industrial applications

Educational Qualifications

To be considered a mathematician, one should have at least a Bachelor of Arts or a Bachelor of Science degree, with major in mathematics. Normally an additional year of training would be required for professional standing but the great need during the national crisis necessitates the use, at a professional level, of all persons holding a bachelor's degree with major in mathematics.

Professional Affiliations

Mathematicians are organized into three learned societies: The American Mathematical Society, the Mathematical Association of America and the Institute of Mathematical Statistics. Membership is a strong indication of professional standing.

Alternate Designation

Since mathematicians are very likely to have specialized in some field of application, it is often true that the designation of the individual may be in terms of that field. Thus, in the field of mathematical physics may be found physicists or mathematicians. Similarly advanced electrical engineering is dependent upon applied mathematics. Applied mathematicians may be employed for advanced work and designated in terms of the fields listed below:

Electrical engineering	Metallurgy
Mechanical engineering	Communications
Chemistry	Astronomy
Biology	Economics
Physics	Ballistics

Industry

Mathematicians are employed in educational institutions, by government both Federal and state, in research institutions, and to some extent in industry.

FROM THE NATIONAL ROSTER ON NEW SELECTIVE SERVICE REGULATIONS

In the MONTHLY, February, 1944, appeared a report pertaining to new regulations on student deferment issued by Selective Service Headquarters. Several communications have been sent by the National Roster to presidents of all institutions of higher education in the country giving additional information upon the new rulings. Some important excerpts from these letters follow.

Letter of January 20. "Graduate students are not covered by Activity and Occupation Bulletin 33-6." (The status of graduate students as teachers is now governed by LBM 115, amended January 6, 1944. This Memorandum states, "A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction.")

Letter of January 27. "Because of the pressing national need for young men in the armed services, the Selective Service System has decided that it can allow the deferment of a total of only 10,000 undergraduate students in the fields of engineering, physics, chemistry, geology and geophysics. The national quota of 10,000 has been distributed by fields as follows: engineering, 6,775; physics, 850; chemistry, 2,250; geology and geophysics, 125. The total number of students presently deferred in these fields is considerably in excess of the 10,000 allowed. The National Roster has no authority in connection with the establishment of overall national quota. Its function is purely administrative. Therefore, it now becomes the obligation of the individual college to determine, within the quota assigned to it, the students for whom deferment may be requested. This should be done in accordance with provisions of Activity and Occupation Bulletin No. 33-6, as amended January 6, 1944.

"The initial quota assigned to your institution has been computed by multiplying the present number of students in Class II and 'Class II pending' (which you recently reported to us) by the ratio of the national quota to the total number of such students reported by all institutions."

THE CURTAILMENT OF THE ARMY COLLEGE PROGRAM

The press notice issued February 18, 1944, by the War Department in regard to the curtailment of the Army Specialized Training Program is reproduced below.

"The shortage of personnel from which the Army is now suffering has led the War Department to drastic decisions during the past week. Because of the inability of the Selective Service to deliver personnel according to schedule, the Army is now short 200,000 men who should have been in uniform before the end of 1943. The increased tempo of offensive operations together with the mounting casualties demanding immediate replacements in the field have created a situation which has necessitated drastic economies in the employment of personnel throughout the United States, and a decision to reduce the soldiers in colleges taking the Army Specialized Training from 145,000 to 35,000. This last measure has been rendered necessary by the imperative requirement at this time for these men who have already had their basic training and a certain amount of specialized training for which their services are now urgently needed.

"After exhausting all other sources, it was determined that the type of trained military personnel needed could be obtained only by decreasing the number of combat units or by drawing from the reservoir of men in ASTP training. It was decided that military necessity required that existing combat units be maintained.

"The 35,000 remaining in the program will be primarily those trainees taking advanced courses in medicine and dentistry, or engineering and include 5,000 pre-induction students. The students withdrawn will be those already basically trained and on active duty. Seventeen-year-olds in the Army Specialized Training Program Reserve will not be affected, nor will this Reserve phase of the program be curtailed.

"The student soldiers now in the Army Specialized Training Program were selected for their high intelligence, adaptability, and potential leadership. They are the type who can be expected to assume the responsibilities of non-commissioned officers and of skilled technicians. Experience to date in this war has demonstrated to the Army that the combat arms, particularly the infantry, need a substantial proportion of men with these qualities to insure continued success in operations. All experience also has shown conclusively that losses are considerably lower in units which have intelligent and aggressive leadership among non-commissioned officers.

"Reassignment from ASTP to other duty before April 1 will be made, so far as military necessity permits, at the completion of a particular training

course or a term in that course. Colleges will be reimbursed for the unexpired portion of contracts covering students withdrawn from ASTP.

"The War Department believes, on the basis of experience, the infusion of thousands of highly intelligent student soldiers into the ground forces, which will see more action as the tempo of our offensive increases, will help to increase our striking power. Consequently, around 80,000 of the men to be transferred from ASTP will be assigned to the Army Ground Forces where the skills and capacity for leadership are now most needed. Most of the remainder will be assigned to other units destined for overseas service. The policy will be to make certain that the skills and the qualities of leadership which these thousands of student soldiers possess are used on assignments where they can function most effectively."

THE ARMY AND NAVY COLLEGE RESERVE PROGRAMS

On March 3, 1944, the Army and the Navy issued a joint press release explaining the expanded Reserve Programs to be instituted in many colleges and universities which have previously participated in the Army and the Navy College Training Programs. To be in the first group of trainees selected for the new Reserve Programs, a student must have made a satisfactory grade upon the "Army-Navy College Qualifying Test" given March 15.

The test of March 15 was the third opportunity given male civilians, who meet the age and academic requirements, to enter the College Training Programs. Those who took the test were required to indicate on the day of the test their preference for the Army Program or that of the Navy. Young men graduated from high school or in their final term and who will have reached their 17th but not their 22nd birthday by July 1, 1944, were permitted to express a preference for the Army; those who will have reached their 17th but not their 19th birthday by July 1 could express a preference for the Navy. Taking the test did not constitute enlistment in any branch of the armed services; that is, having taken the test, a student is not obligated to enter the Program if he is accepted.

Seventeen-year-olds who qualified on the test, and who expressed Navy preference and are accepted, will receive the same training as other students in the Navy College Program, which is designed to provide officers for the Navy. These seventeen-year-olds will be enlisted in Class V-12, U. S. Naval Reserve, and will not be inducted as in the case of 18- and 19-year-olds. Like other students in the Navy College Program, they will be placed in uniform, with pay, under military discipline.

Successful contestants in the March 15 test, who will be less than 17 years and nine months old on July 1, 1944, and who have stated an Army preference, will form the group who enter the AST Reserve Program. This age restriction will, except in the event of an unforeseen military emergency, assure a minimum of six months of intensive academic work at the college level before the individual may be called to active Army duty for his basic military training. Those who

are less than 17 years and six months old on July 1, 1944, will receive, subject to the same conditions, at least nine months of academic work under the AST Reserve Program.

It is hoped that a sufficient number of seventeen-year-olds will be found qualified so that a substantial expansion of the AST Reserve Program can be made. This would tend to replace some of the loss in men trained at the college level occasioned by the recent return to active troop duty of many ASTP trainees.

The majority of eligible students who qualify for the military scholarships offered under the AST Reserve Program will be assigned to one of the following curricula: (1) applied sciences; (2) chemical and biological sciences preparatory to advanced medical and dental studies; (3) mathematics and physics. Assignment will be based on the qualifications and aptitudes of the individual. A limited number of specially qualified men may be assigned to a new foreign language course.

A limited number of men between 17 years, nine months old and 22 years old on July 1, 1944, who took the March 15 Army-Navy College Qualifying Test, will be selected for the regular Army Specialized Training Program after their induction into the Army and following their basic military training. In general, those selected will be men who have had prior academic training which qualifies them for advanced engineering and language courses of the Army Specialized Training Program. These men will be selected at the reception centers and will be further screened by a Specialized Training and Reassignment (STAR) board upon completion of their military training.

Students accepted for the regular Navy College Program will attend colleges and universities under contract to the Navy for varying numbers of 16-week terms depending on the type of course they pursue. Prospective deck officers for the Navy will receive four terms of college training in sixteen months after which they will pursue a four months' course at a Naval Reserve Midshipman's School before being commissioned as ensigns. Others will take courses of greater length leading to commissions as physicians, dentists, chaplains, engineers, supply officers and other specialists.

THE NEED FOR WOMEN IN THE ARMED SERVICES

The armed forces are making a new call for recruits among American college women who have had specialized training. Women with training in mathematics are needed in every branch of service.

The Army reports that on January 1, 1944, there were 62,859 officers and enlisted women in the WAC. Its authorized strength is 200,000. A college woman with technical training stands a good chance of being selected as an officer candidate. Moreover, it is now possible to apply for a specific type of work or for a specific station assignment.

At the present time there are approximately 50,000 WAVES, and recruiting drives are expected to double this number by the end of 1944. The Navy is now selecting college graduates more on the basis of their ability to do certain jobs

than on their personal qualifications. At the present time a search is being conducted especially for women trained in aerology, mathematics, physics, navigation, supply, or medicine.

The SPARS are seeking college women for officer candidates who have had three or more years of work experience. Women capable of supervising enlisted personnel are urgently needed. Opportunities are especially good for pay and supply officers, and for those having the background to do work in communications and ordnance.

The MARINES need women in the 25 to 35 age group for officer candidates. Those who can qualify as specialists are urged to apply.

In addition to the important service which a woman can render her country by enrolling in a branch of the armed forces, she will become eligible for the same postwar benefits as those available to men who are veterans of this war. In particular, valuable educational opportunities will probably be made available.

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS, Topeka, April 15, 1944
KENTUCKY, Lexington, April 29, 1944
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN
MINNESOTA, St. Paul, May 6, 1944
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

JUST PUBLISHED

W. L. HART'S

Plane and Spherical Trigonometry

WITH APPLICATIONS

An efficient, concise presentation of the fundamentals of plane trigonometry with strong emphasis on numerical applications to technical war problems, and a well-rounded treatment of spherical trigonometry with extensive applications that provide an excellent background for the study of celestial navigation. *224 pages. $2.00. With tables, 352 pages. $2.25*

Previously published ~

Plane Trigonometry, WITH APPLICATIONS

177 pages. $1.75. With tables, 304 pages. $2.00

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

TRIG PROBLEMS ARE EASY...

with these new Slide Rules

The improved arrangement of the scales on these new Log Log Duplex Trig* and Log Log Duplex* Decitrig Slide Rules greatly simplifies the handling of trig problems. The trig scales refer directly to the CD scales so that trigonometrical functions can be used as factors without having their numerical values determined.

*Trade Mark

KEUFFEL & ESSER CO.

NEW YORK · HOBOKEN, N. J.

Chicago St. Louis San Francisco Los Angeles Detroit Montreal

McGraw-Hill Books of Timely Interest

MODERN OPERATIONAL MATHEMATICS IN ENGINEERING

By RUEL V. CHURCHILL, University of Michigan. In press—ready in April

Partial differential equations of engineering and Laplace transforms are the two principal topics treated. Problems in ordinary differential equations and other types of problems are included. The operational properties of the Laplace transformation are derived and carefully illustrated and are used to solve problems of various kinds.

METHODS IN ADVANCED CALCULUS

By PHILIP FRANKLIN, Massachusetts Institute of Technology. In press—ready in April

Covers those aspects of advanced calculus that are most needed in applied mathematics. The first five chapters deal with Taylor's series, partial differentiation and applications to space geometry, and integration. The later chapters deal with specific higher functions, Fourier series, differential equations, vector analysis, and the calculus of variations.

MATHEMATICAL AND PHYSICAL PRINCIPLES OF ENGINEERING ANALYSIS

By WALTER C. JOHNSON, Princeton University. 343 pages, $3.00

Presents the essential physical and mathematical principles and methods of attack that underlie the analysis of many practical engineering problems. The book emphasizes fundamentals and physical reasoning, and devotes considerable attention to methods of attack, the use of assumptions, procedures in setting up equations, the use of mathematics as a tool in accurate and quantitative reasoning, and the physical interpretation of mathematical results.

VECTOR AND TENSOR ANALYSIS

By HOMER V. CRAIG, University of Texas. 434 pages, $3.50

Provides a fairly rigorous, yet clear and comprehensive introduction to the important subject of vector and tensor analysis. The treatment is especially adapted to those who do not have a thorough knowledge of modern advanced calculus. Transformation and invariantive aspects are emphasized.

DIFFERENTIAL EQUATIONS

By RALPH P. AGNEW, Cornell University, 341 pages, $3.00

Written for students with a reasonably good knowledge of elementary calculus, so that they may master the techniques by which differential equations are obtained and solved and by which the solutions are applied.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

Always

BASIC MARINE NAVIGATION

By BART J. BOK and FRANCES W. WRIGHT, Harvard University

A simple, yet complete, treatment of coastwise and offshore navigation. The KIT OF PRACTICE MATERIALS furnished with the text affords, with the text, a complete and effective unit for the study of navigation.

CONCISE SPHERICAL TRIGONOMETRY

By JACQUES REDWAY HAMMOND, U. S. Naval Academy, Annapolis

Designed for a two-or three-hour semester course especially directed to those who need spherical trigonometry as background for navigation, meteorology, cartography, and like subjects. Solid geometry and plane trigonometry are reviewed.

An Introduction to SPHERICAL TRIGONOMETRY

By FRANK LOXLEY GRIFFIN, Reed College

A brief and inexpensive treatment of spherical trigonometry covering the points essential for the efficient solving of spherical triangles.

HOUGHTON MIFFLIN COMPANY

Boston New York Chicago Dallas Atlanta San Francisco

Raymond W. Brink's

A FIRST YEAR OF COLLEGE MATHEMATICS

"Remarkable for its adequate treatment of so many topics."

—Hammond Pride, New York University.—

655 pages .....price $3.75

with SPHERICAL TRIGONOMETRY, price $4.00

D. APPLETON-CENTURY COMPANY

35 West 32nd St. New York 1, N.Y.

Crofts

ANALYTIC GEOMETRY

By K. B. Patterson *and* A. O. Hickson, *Duke University*

A brief text which can be covered in from forty to fifty class periods. Teachers are praising its compactness and clarity of explanation, its logical arrangement of topics, and the variety of exercise material—all in an attractive format.

Published in February, 187 pages, $2.10

ARITHMETIC FOR ADULTS

A Review of Elementary Mathematics

By Aaron Bakst

A sense of urgency, even of excitement about the subject, pervades this new book by the author of *Mathematics, Its Magic and Mastery,* and of *Approximate Computations* (Twelfth Yearbook of the National Council of Teachers of Mathematics).

Dr. Bakst starts from scratch assuming that the reader has remembered little from his earlier study, and presents clearly and in logical progression *arithmetical processes* and their *practical applications* in every-day living, from the simplest "1-2-3" to the use of the slide rule.

We suggest that teachers consider this new book as a text for review of arithmetic on the college freshman or the high school senior level, for correspondence and extension classes, for courses in the teaching of arithmetic, and for recommendation to adults who could profit from a mastery of this often-forgotten subject in their day-to-day affairs.

Published April 10th, 319 pages, $2.00

FIRST YEAR COLLEGE MATHEMATICS

By C. C. Richtmeyer *and* J. W. Foust, *Central Michigan College of Education*

Starting with a review of elementary algebra and covering algebra, trigonometry, plain analytics, and an introduction to the fundamental concepts of differentiation and integration, this book gives an excellent foundation course, unified by the *function* principle. Contains 500 self-tests, 4,000 problems, numerous illustrative examples.

461 pages, $3.25

For further information, write to:

F. S. CROFTS & CO.

101 Fifth Avenue New York 3, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

ALWAYS

5

HNOLOGY DEPT:

PUBLIC LIBRARY
JUN 1 1944
DETROIT

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 51 NUMBER 5

CONTENTS

Deformations of Plane Rectilinear Complexes . . . S. S. CAIRNS 247
Associated Frequency Distributions in Biometry . . R. H. COLE 252
On the Densest Packing of Circles . . B. SEGRE AND K. MAHLER 261
An Application of a Theorem of Sylvester . . . L. R. WILCOX 270
Clubs and Allied Activities . . . . . H. D. LARSEN 274
Discussions and Notes . . . . . W. V. LOVITT, ROY DUBISCH 276
Recent Publications 279
Problems and Solutions 284
News and Notices 294
War Information 298
Salaries and Teaching Loads in College Training Programs . . . 298
Educational Credit for Military Experience 300
Future of the Navy V-12 Programs 302
Notes on the Training Programs 302
Selective Service Regulations Concerning Students 303
The Mathematical Association of America 304
New Members 304
Northern California Section Meeting 306
Calendar of Future Meetings 308

MAY 1944